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Elliptic and parabolic equations with measurable nonlinearities in nonsmooth domains

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김유찬

Elliptic and parabolic equations with measurable nonlinearities in nonsmooth domains

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Youchan Kim

Dissertation Director : Professor Sun-Sig Byun

Department of Mathematical Sciences

Seoul National University

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Abstract

We study elliptic and parabolic equations with measurable nonlinearities in nonsmooth domains. We establish an optimal global $W^{1,p}$ estimate under the condition that the associated nonlinearity is allowed to be merely measurable in one variable but has a sufficiently small BMO semi-norm in the other variables, while the underlying domain is sufficiently flat in the Reifenberg sense that the boundary of the domain is locally trapped between two narrow strips.

Key words: Calderón-Zygmund estimate, nonlinear elliptic equation, nonlinear parabolic equation, measurable nonlinearity, Reifenberg domain

Student Number: 2006-22887

Contents

Chapter 1

Introduction

This dissertation concerns with an optimal L^p-regularity theory, so called Calderón-Zygumnd type regularity theory. Calderoón-Zygmund theory investigates higher integrability of the gradient or the Hessian of solutions to elliptic and parabolic partial differential equations. When dealing with Calderón-Zygmund type regularity theory, the regularity condition on nonlinearity a and the boundary $\partial\Omega$ should be carefully considered, and the problem finding the minimal regularity requirement on a and $\partial\Omega$ is a classical problem in the regularity theory.

Let u be the weak solution to the following nonlinear elliptic equation

$$\begin{cases} \operatorname{div} a(Du, x) & = & \operatorname{div} F & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ ($n \geq 2$), under the assumption that $a(\xi, x)$ is the Carathéordory nonlinearity $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following ellipticity and growth conditions:

$$\begin{cases} |a(\xi, x)| & \leq & \Lambda|\xi|, \\ |D_\xi a(\xi, x)| & \leq & \Lambda, \\ \langle D_\xi a(\xi, x)\zeta, \zeta\rangle & \geq & \lambda|\zeta|^2, \end{cases} \tag{1.2}$$

for a.e. $x \in \mathbb{R}^n$, for every $\xi, \zeta \in \mathbb{R}^n$ and for some constants $0 < \lambda \leq \Lambda$.

If the nonlinearity a and the boundary $\partial\Omega$ sufficiently regular enough, then the weak solution u of (1.1) satisfies

$$\|Du\|_{L^p(\Omega;\mathbb{R}^n)} \leq C\|F\|_{L^p(\Omega;\mathbb{R}^n)}, \quad p \in [2, \infty), \tag{1.3}$$

where $C > 0$ is independent of u and F. This estimate implies that

$$F \in L^p(\Omega;\mathbb{R}^n) \quad \Longrightarrow \quad Du \in L^p(\Omega;\mathbb{R}^n), \quad p \in [2,\infty). \tag{1.4}$$

The research topic of establishing Calderón-Zygmund type estimates (1.3) for elliptic problems, in particular with discontinuous coefficients or on nonsmooth domains, has been a classical and rich one. For linear equations, namely, when $a(\xi,x) = A(x)\xi$ for some positive definite $n \times n$ matrix $A(x)$, the fundamental estimate like (1.3) was obtained by many authors for all $p \in (1,\infty)$. Calderón and Zygmund proved (1.3) when $A = I_n$ in [14], and their results were extended to the case of uniformly continuous coefficients by Morrey in [29], Simader in [37], and Campanato in [15]. For discontinuous coefficients, the cases that A is in VMO(vanishing mean oscillation) and A has a small deviation from being regular were widely considered. Di Fazio investigated when A is in VMO in [20]. Later, the case when $A - I_n$ is small in L^n was done by Caffarelli and Peral in [13] for the interior case, and the case when A has small BMO(bounded mean oscillation) and on very flat domains were proved by Byun and Wang in [7]. As far as we are concerned with nonlinear elliptic equations, the condition that $\frac{a(\xi,x)}{|\xi|}$ has small BMO in the variable x, uniformly in $\xi \neq 0$ and $\partial\Omega$ is sufficiently flat is the most general condition on the associated coefficients to have the gradient estimate (1.3), see [8, 30, 31, 32, 33, 35].

The results in the previous paragraph had the regularity assumption on the nonlinearity $a(\xi,x)$ that $a(\xi,x)$ is sufficiently regular in the all spatial directions. This assumption is important because if $a(\xi,x)$ is allowed to be merely measurable in two spatial variables, then $W^{1,p}$-regularity fails in general, see the classical example of Meyers in [28]. When $a(\xi,x)$ is allowed to be merely measurable in one spatial variable, [16] showed interior Lipschitz regularity for linear elliptic systems. Recently, Calderón-Zygmund type estimates (1.3) are obtained on linear elliptic problems with merely measurable coefficients in nonsmooth domains by Dong and Kim [17] and Byun and Wang [9] independently, and have influenced later results [18, 4, 5]. As far as we are concerned in the literature, little is known for elliptic equations with measurable nonlinearity category which allows one merely measurable spatial variable on $a(\xi,x)$. The reason is that the previous works in [9, 17] heavily relied on the linear structure of the equations and cannot be generally extended to the nonlinear problems used there.

Our improvement shall be made by using a new approach based on De

Giorgi theory, in order to obtain Lipschitz regularity on limiting equations. Especially on the boundary points, we use a new method based on Poincaré's inequality and Campanato type embedding theorem to control the normal derivatives, instead of using the standard barrier argument or the Sobolev embeddings. With the Lipschitz regularity on limiting equations, we prove global $W^{1,p}$ estimate (1.3). More precisely, we show that the global $W^{1,p}$-estimate holds true for the weak solution of (1.1) when $p \in [2, \infty)$, if for each point and for each scale the nonlinearity $a(\xi, x)$ is only measurable in one variable and are averaged in the sense of small BMO with respect to the remaining $n-1$ variables, while the boundary $\partial\Omega$ can be trapped into two hyperplanes depending on the scale chosen. For each point and for each scaling, we compare the weak solution of (1.3) to a weak solution of a limiting equation. Then we use a classical harmonic analysis tools such as maximal function to show that $u \in W^{1,p}(\Omega)$.

Now, we consider parabolic equations. We denote by $\partial_p\Omega_T = \partial\Omega \times [0, T] \cup \Omega \times \{t = 0\}$ to mean the parabolic boundary of the parabolic cylinder $\Omega_T = \Omega \times (0, T]$. Let u be the weak solution to the following nonlinear parabolic equation

$$\begin{cases} u_t - \operatorname{div} a(Du, x, t) &= \operatorname{div} F \quad \text{in } \Omega_T, \\ u &= 0 \qquad\quad \text{on } \partial_p\Omega_T, \end{cases} \tag{1.5}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, under the assumption that $a(\xi, x, t)$ is the Carathéordory nonlinearity $a : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ satisfying the following ellipticity and growth conditions:

$$\begin{cases} |a(\xi, x, t)| &\leq \Lambda|\xi|, \\ |D_\xi a(\xi, x, t)| &\leq \Lambda, \\ \langle D_\xi a(\xi, x, t)\zeta, \zeta\rangle &\geq \lambda|\zeta|^2, \end{cases} \tag{1.6}$$

for a.e. $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, for every $\xi, \zeta \in \mathbb{R}^n$ and for some constants $0 < \lambda \leq \Lambda$.

If the nonlinearity a and the boundary $\partial\Omega$ sufficiently regular enough, then the weak solution u of (1.5) satisfies

$$\|Du\|_{L^p(\Omega_T;\mathbb{R}^n)} \leq C\|F\|_{L^p(\Omega_T;\mathbb{R}^n)}, \quad p \in [2, \infty), \tag{1.7}$$

where $C > 0$ is independent of u and F. This estimate implies that

$$F \in L^p(\Omega_T; \mathbb{R}^n) \quad \Longrightarrow \quad Du \in L^p(\Omega_T; \mathbb{R}^n), \quad p \in [2, \infty). \tag{1.8}$$

For the linear case that $a(\xi, x, t) = A(x, t)\xi$ with $A(x, t)$ being positive definite, the estimate (1.7) was obtained in [10] under the conditions that the principal coefficients have small BMO(bounded mean oscillation) and $\partial\Omega$ is sufficiently flat. The result [10] is extended to nonlinear parabolic equations in [8] with the condition that $\frac{a(\xi,x,t)}{|\xi|}$ has small BMO in the variable (x, t), uniformly in $\xi \neq 0$ and $\partial\Omega$ is sufficiently flat. Also the result in [10] was extended to the measurable coefficients in [6, 17] under the conditions that that the coefficients have no regularity requirement in one of space variables but have small BMO in the other variables.

We prove (1.7), when for each point and for each scale the nonlinearity $a(\xi, x, t)$ is only measurable in one variable and are averaged in the sense of small BMO with respect to the remaining $n - 1$ and t variables, while the boundary $\partial\Omega$ can be trapped into two hyperplanes depending on the scale chosen. The approach is similar to the case of elliptic equations, but we need a more delicate argument for Lipschitz regularity for limiting equations because there is a new term u_t in the equation (1.5).

Chapter 2

Elliptic equations

2.1 Definitions and main result

We introduce the following notations:

1. $x = (x^1, x^2, \cdots, x^n) = (x^1, x')$, $\xi = (\xi_1, \cdots, \xi_n)$ and $\zeta = (\zeta_1, \cdots, \zeta_n)$.
2. $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$, $B_\rho(y) = B_\rho + y$, $B_\rho^+ = B_\rho \cap \{x : x^1 > 0\}$, $B_\rho^+(y) = B_\rho^+ + y$, $B_\rho^- = B_\rho \cap \{x : x^1 < 0\}$, $B_\rho^-(y) = B_\rho^- + y$.
3. $\Omega_\rho(y) = B_\rho(y) \cap \Omega$, $\partial_w \Omega_\rho(y) = \partial\Omega \cap B_\rho(y)$.
4. $T_\rho = B_\rho \cap \{x^1 = 0\}$.
5. $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$.
6. For $f \in L^1(U)$, $(f)_U = \fint_U f \, dx = \frac{1}{|U|} \int_U f \, dx$ is the integral average over a bounded set U in $\mathbb{R}^n$.

Assume $a(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéordory vector field, that is

$$\begin{cases} a(\xi, x) \text{ is measurable in } x \in \mathbb{R}^n \text{ for every } \xi \in \mathbb{R}^n, \\ a(\xi, x) \text{ is } C^1\text{-regular in } \xi \in \mathbb{R}^n \text{ for a.e. } x \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

We impose the following ellipticity and growth conditions:

$$\begin{cases} |a(\xi, x)| & \leq & \Lambda|\xi|, \\ |D_\xi a(\xi, x)| & \leq & \Lambda, \\ \langle D_\xi a(\xi, x)\zeta, \zeta\rangle & \geq & \lambda|\zeta|^2, \end{cases} \tag{2.2}$$

for a.e. $x \in \mathbb{R}^n$, for every $\xi,\ \zeta \in \mathbb{R}^n$ and for some constants $0 < \lambda \leq \Lambda$. Then it is readily check from (2.2) that

$$a(0,x) = 0 \text{ and } \langle a(\xi,x) - a(\zeta,x), \xi - \zeta\rangle \geq \lambda|\xi - \zeta|^2. \tag{2.3}$$

With the assumptions (2.1) and (2.2), we consider the following Dirichlet problem

$$\begin{cases} \text{div } a(Du,x) & = & \text{div } F & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $F \in L^2(\Omega;\mathbb{R}^n)$ is a given vector-valued function.

We consider a weak solution in a classical Sobolev space $W_0^{1,2}(\Omega)$, which means

$$\int_\Omega \langle a(Du,x), D\varphi\rangle\, dx = \int_\Omega \langle F, D\varphi\rangle\, dx, \text{ for all } \varphi \in W_0^{1,2}(\Omega).$$

To measure the oscillation of $a(\xi,x) = a(\xi,x^1,x')$ in x'-variables with respect to $\bar{a}(\xi,x^1)$, we define a function $\theta(a,\bar{a})(x)$ as below

$$\theta(a,\bar{a})(x) = \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|a(\xi,x^1,x') - \bar{a}(\xi,x^1)|}{|\xi|}, \tag{2.5}$$

for any $\bar{a}(\xi,x^1)$ satisfying (2.1) and (2.2).

We introduce the main assumptions on a and Ω.

Definition 2.1.1. We say that $(\Omega, a(\xi,x))$ is (δ,R_0)-vanishing of codimension 1 if the following holds.

1. For every ball $B_r(x_0) \subset \Omega$ with $r \in (0,R_0]$, there exists a coordinate system depending on x_0 and r, whose variables we still denote by $x = (x^1,x')$, so that in this new coordinate system x_0 is the origin and

$$⨍_{B_r} |\theta(a,\bar{a})(x)|^2\, dx \leq \delta^2,$$

for some $\bar{a}(\xi,x^1)$ satisfying (2.1) and (2.2).

2. For every point $x_0 \in \partial\Omega$ with $r \in (0,R_0]$, there exists a coordinate system depending on x_0 and r, whose variables we still denote by $x = (x^1,x')$, so that in this new coordinate system x_0 is $-\delta r e_1$,

$$B_r^+ \subset B_r \cap \Omega \subset B_r \cap \{(x^1,x') : x^1 > -2\delta r\},$$

and
$$\fint_{B_r} |\theta(a,\bar{a})(x)|^2 \, dx \le \delta^2,$$
for some $\bar{a}(\xi, x^1)$ satisfying (2.1) and (2.2).

Remark 2.1.2. *Throughout this paper $0 < \delta < \frac{1}{8}$ is a small constant to be determined later so that the main result Theorem 2.1.5 holds for $2 \le p < \infty$. On the other hand, R_0 can be any number which is bigger than 1 by the scaling invariance of the problem (2.4), see Lemma 2.2.1 in the next section.*

Remark 2.1.3. *When we defined (δ, R_0)-vanishing of codimension 1 in the earlier results, we focused on the small BMO condition and the term $\bar{a}(\xi, x^1)$ in Definition 2.1.1 was specified as the mean average over x'-variables by using x_0 and r, say $\bar{a}(\xi, x^1) = \fint_{|z'-x_0'|<r} a(\xi, x^1, z') \, dz'$. So if $a(\xi, x)$ has a small BMO semi-norm in the category of measurable nonlinearities, then $a(\xi, x)$ also satisfies our new definition of (δ, R_0)-vanishing of codimension 1 in Definition 2.1.1.*

Remark 2.1.4. *From the above definition, one see that there is no regularity assumption on the nonlinearity $a(\xi, x)$ with respect to x^1 variable, and so there might be big jumpings of the nonlinearity $a(\xi, x)$ along the x^1 variable while the nonlinearity a is being averaged along the x' variables. Note that if $(\Omega, a(\xi, x))$ is (δ, R_0)-vanishing of codimension 1, then Ω is (δ, R_0)-Reifenberg flat domain, see [7, 40] for the concept of being δ-Reifenberg flat.*

We now state the main theorem of this paper.

Theorem 2.1.5. *Assume that $F \in L^p(\Omega; \mathbb{R}^n)$ for some $p \in [2, \infty)$. Then there exists a small $\delta = \delta(n, \lambda, \Lambda, p, |\Omega|) > 0$ such that if $(\Omega, a(\xi, x))$ is (δ, R_0)-vanishing of codimension 1, then the unique weak solution $u \in W_0^{1,2}(\Omega)$ of (2.4) satisfies $Du \in L^p(\Omega; \mathbb{R}^n)$ with the following estimate*
$$\int_\Omega |Du|^p \, dx \le C \int_\Omega |F|^p \, dx, \tag{2.6}$$
where the constant C depending only on n, λ, Λ, p and $|\Omega|$.

Remark 2.1.6. *For the sake of convenience and simplicity, we employ the letters $C > 0$ to denote any constants which can be explicitly computed in terms of known quantities such as n, λ, Λ and p. Thus the exact value denoted by C may change from line to line in a given computation.*

2.2 Preliminaries

We start this section with the following scaling and normalization which we use in this chapter.

Lemma 2.2.1. *For each $r, M > 0$, let us define the scaling and renormalization maps:*

$$\tilde{a}(\xi, x) = \frac{a(M\xi, rx)}{M}, \ \tilde{u}(x) = \frac{u(rx)}{rM}, \ \tilde{F}(x) = \frac{F(rx)}{M}, \ \tilde{\Omega} = \left\{ \frac{x}{r} : x \in \Omega \right\}.$$

Then we have

1. *If $u \in W_0^{1,2}(\Omega)$ is the weak solution of*

   $$\text{div } a(Du, x) = \text{div } F \text{ in } \Omega,$$

   then $\tilde{u} \in W_0^{1,2}(\tilde{\Omega})$ is also the weak solution of

   $$\text{div } \tilde{a}(D\tilde{u}, x) = \text{div } \tilde{F} \text{ in } \tilde{\Omega}.$$

2. *Suppose that $(\Omega, a(\xi, x))$ is (δ, R_0)-vanishing with codimension 1 for constants λ and Λ. Then $(\tilde{\Omega}, \tilde{a}(\xi, x))$ is $\left(\delta, \frac{R_0}{r}\right)$-vanishing with codimension 1 for the same constants λ and Λ.*

The following lemma is an elementary property of integral averages.

Lemma 2.2.2. *Let $F \in L^2(U; \mathbb{R}^m)$ for some $m \geq 1$. Then we have*

$$\fint_U |F - (F)_U|^2 \, dx \leq \fint_U |F - \xi|^2 \, dx,$$

for every $\xi \in \mathbb{R}^m$.

For $G : U \to \mathbb{R}^m$, define $[G]_{C^{0,\alpha}(U)}$ as the Hölder semi-norm, which is

$$[G]_{C^{0,\alpha}(U)} = \sup_{x,y \in U, \ x \neq y} \frac{|G(x) - G(y)|}{|x - y|^\alpha}.$$

The next lemma is the well known Campanato type embedding. We will use Campanato type embedding for the Lipschitz regularity on the limiting equations in Section 2.3.

Lemma 2.2.3. *[22, Theorem 2.9] Assume $G \in L^2(B_r;\mathbb{R}^m)$ for some $m \geq 1$. Define*

$$[G]^2_{\mathcal{L}^{2,\kappa}(B_r)} = \sup_{x_0\in B_r,\ 0<\rho<diam(B_r)} \frac{1}{\rho^{n+2\kappa}} \int_{B_r\cap B_\rho(x_0)} \left|G - (G)_{B_r\cap B_\rho(x_0)}\right|^2\, dx < \infty.$$

If $0 < \kappa \leq 1$, then we have

$$\|G\|_{L^\infty(B_r)} + r^\kappa [G]_{C^{0,\kappa}(B_r)} \leq C\left[\left(\fint_{B_r} |G|^2\, dx\right)^{\frac{1}{2}} + r^\kappa [G]_{\mathcal{L}^{2,\kappa}(B_r)}\right],$$

for some positive constant $C = C(n,\kappa)$.

Note that $diam(B_r) = 2r$. By using Lemma 2.2.3, we have the following lemma.

Lemma 2.2.4. *Assume that $G \in L^2(B_{3r};\mathbb{R}^m)$ for some $m \geq 1$. Suppose*

$$\fint_{B_\rho(x_0)} \left|G - (G)_{B_\rho(x_0)}\right|^2\, dx \leq C\left(\frac{\rho}{r}\right)^{2\alpha} \fint_{B_{2r}(x_0)} |G|^2\, dx, \tag{2.7}$$

holds for any $x_0 \in B_r$, $0 < \rho < 2r$ and for some $\alpha = \alpha(n,\lambda,\Lambda) \in (0,1)$. Then we have

$$\|G\|_{L^\infty(B_r)} \leq C\left(\fint_{B_{3r}} |G|^2\, dx\right)^{\frac{1}{2}}. \tag{2.8}$$

Proof. For any $\rho \in (0,2r)$, we have from Lemma 2.2.2 that

$$\begin{aligned}\int_{B_r\cap B_\rho(x_0)} \left|G - (G)_{B_r\cap B_\rho(x_0)}\right|^2\, dx &\leq \int_{B_r\cap B_\rho(x_0)} \left|G - (G)_{B_\rho(x_0)}\right|^2\, dx \\ &\leq \int_{B_\rho(x_0)} \left|G - (G)_{B_\rho(x_0)}\right|^2\, dx.\end{aligned} \tag{2.9}$$

Thus we have

$$
\begin{aligned}
[G]^2_{L^{2,\alpha}(B_r)} &= \sup_{x_0\in B_r,\ 0<\rho<2r} \frac{1}{\rho^{n+2\alpha}} \int_{B_r\cap B_\rho(x_0)} \left|G-(G)_{B_r\cap B_\rho(x_0)}\right|^2 \, dx \\
&\le \sup_{x_0\in B_r,\ 0<\rho<2r} \frac{1}{\rho^{n+2\alpha}} \int_{B_\rho(x_0)} \left|G-(G)_{B_\rho(x_0)}\right|^2 \, dx \\
&\le \sup_{x_0\in B_r} \frac{C}{r^{2\alpha}} \fint_{B_{2r}(x_0)} |G|^2 \, dx \\
&\le \frac{C}{r^{2\alpha}} \fint_{B_{3r}} |G|^2 \, dx.
\end{aligned} \tag{2.10}
$$

By using (2.10), Lemma 2.2.3 implies that

$$
\|G\|_{L^\infty(B_r)} \le C\left[\left(\fint_{B_r} |G|^2 \, dx\right)^{\frac{1}{2}} + r^\alpha [G]_{L^{2,\alpha}(B_r)}\right] \le C\left(\fint_{B_{3r}} |G|^2 \, dx\right)^{\frac{1}{2}}.
$$

□

The next lemma which will be used in proving the desired Lipschitz regularity.

Lemma 2.2.5. *Let $\nu_0 = \frac{\lambda}{2\Lambda} \le \frac{1}{2}$. Under the assumption (2.1) and (2.2), we have*

$$
\nu_0|\xi| \le 2|(\nu_0\lambda^{-1}a^1(\xi,x),\xi')| \le 4|\xi|, \tag{2.11}
$$

for a.e. $x\in\mathbb{R}^n$ and for every $\xi=(\xi_1,\xi')\in\mathbb{R}^n$.

Proof. Fix any $\xi=(\xi_1,\xi')\in\mathbb{R}^n$ and a.e. $x\in\mathbb{R}^n$ such that $a(\xi,x)$ is C^1-regular in $\xi\in\mathbb{R}^n$ by using (2.1). The second inequality is clear from (2.2). So we only prove the first inequality. We first claim that

$$
|a^1(\xi_1,0',x)| \ge \lambda|\xi_1| \quad (\xi\in\mathbb{R}^n). \tag{2.12}
$$

If $\xi_1=0$, then (2.12) holds trivially. If $\xi_1\neq 0$, then we take $\xi=(\xi_1,0')\in\mathbb{R}^n$ and $\zeta=0$ in (2.3) to find that

$$
a^1(\xi_1,0',x)\xi_1 \ge \lambda|\xi_1|^2 > 0,
$$

which implies (2.12). From (2.2), we have

$$\begin{aligned}
&|a^1(\xi_1,\xi',x)-a^1(\xi_1,0',x)| \\
&\quad\le \left|\int_0^1 \frac{d}{ds}a^1(s\xi+(1-s)(\xi_1,0'),x)\ ds\right| \\
&\quad\le \int_0^1 |D_\xi a^1(s\xi+(1-s)(\xi_1,0'),x)||(0,\xi')|\ ds \\
&\quad\le \Lambda|\xi'|.
\end{aligned}\tag{2.13}$$

In view of (2.12) and (2.13), we have

$$|a^1(\xi,x)|\ge|a^1(\xi_1,0',x)|-|a^1(\xi,x)-a^1(\xi_1,0',x)|\ge\lambda|\xi_1|-\Lambda|\xi'|.\tag{2.14}$$

Then by a direct calculation, we discover from (2.14) that

$$\frac{\nu_0}{\lambda}|a^1(\xi,x)|+|\xi'|\ge\nu_0|\xi_1|+\left[1-\frac{\nu_0\Lambda}{\lambda}\right]|\xi'|\ge\nu_0|\xi_1|+\frac{|\xi'|}{2}\ge\nu_0[|\xi_1|+|\xi'|]\ge\nu_0|\xi|.$$

Since $(a+b)^2\le 2a^2+2b^2$ for any $a,b\ge 0$, the above inequality implies the first inequality in (2.11). □

We use the standard regularity results which can be obtained by De Giorgi theory for linear elliptic equations. Fix any $R>0$ and consider a linear elliptic equation

$$D_i(a_{ij}(x)D_jv)=0 \text{ in } B_R,\tag{2.15}$$

with ellipticity and growth conditions:

$$\begin{cases} |a_{ij}(x)|\le\Lambda, \\ a_{ij}(x)\zeta_i\zeta_j\ge\lambda|\zeta|^2, \end{cases}\tag{2.16}$$

for a.e. $x\in B_R$ and every $\zeta\in\mathbb{R}^n$.

Then we have the following better regularity, see for instance [22].

Lemma 2.2.6. *Let v be a weak solution of (2.15). Then we have $Dv\in$*

$L^2_{loc}(B_R)$ and the following estimates

$$\begin{cases} \fint_{B_\rho} |v-(v)_{B_\rho}|^2 \, dx \le C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{B_r} |v|^2 \, dx, \\ \int_{B_\rho} |Dv|^2 \, dx \le \frac{C}{(r-\rho)^2} \int_{B_r} |v-(v)_{B_r}|^2 \, dx, \end{cases}$$

for any $0<\rho<r\le R$ and for some constant $\alpha=\alpha(n,\lambda,\Lambda)\in(0,1)$.

We will combine the classical Hardy-Littlewood maximal function, a Vitali covering lemma and the standard arguments of measure theory.

Definition 2.2.7. The Hardy-Littlewood maximal function $\mathcal{M}f$ of a locally integrable function f defined in $\mathbb{R}^n$ is a function such that

$$(\mathcal{M}f)(x) = \sup_{\rho>0} \fint_{B_\rho(x)} |f(y)| \, dy.$$

If f is defined in a bounded subset E of $\mathbb{R}^n$, we define the restricted maximal function $\mathcal{M}_E f$ by

$$\mathcal{M}_E f = \mathcal{M}\left(f\chi_E\right),$$

where χ_E is the standard characteristic function on E. We will drop the index E in $\mathcal{M}_E f$, if E is understood clearly in the context.

The basic properties for the Hardy-Littlewood maximal function are the followings.

Lemma 2.2.8. *[39] If $f=f(x)\in L^p(\mathbb{R}^n)$ with $1<p\le\infty$, then $\mathcal{M}f\in L^p(\mathbb{R}^n)$ and*

$$\frac{1}{C}\|f\|_{L^p} \le \|\mathcal{M}f\|_{L^p} \le C\|f\|_{L^p}, \tag{2.17}$$

for some constant $C=C(n,p)$. If $f\in L^1(\mathbb{R}^n)$, then

$$|\{x\in\mathbb{R}^n : (\mathcal{M}f)(x)>\alpha\}| \le \frac{C}{\alpha}\int |f(y)| \, dy, \tag{2.18}$$

for some constant $C=C(n)$.

We will use the following version of the Vitali covering lemma.

Lemma 2.2.9. *[7] Let E and F be measurable sets with $E \subset F \subset \Omega$. Assume that Ω is $(\delta, 1)$-Reifenberg flat for some small $\delta > 0$. Assume further that there exists a small $\epsilon > 0$ such that*

$$|E| < \epsilon |B_1|,$$

and for all $x \in \Omega$ and for all $r \in (0, 1]$ with $|E \cap B_r(x)| \geq \epsilon |B_r(x)|$,

$$B_r(x) \cap \Omega \subset F.$$

Then we have

$$|E| \leq \left(\frac{10}{1 - \delta} \right)^n \epsilon |F|.$$

We use the following standard arguments of measure theory.

Lemma 2.2.10. *[12] Assume that f is a nonnegative and measurable function in $\mathbb{R}^n$. Assume further that f has a compact support in a bounded subset E of $\mathbb{R}^n$. Let $\theta > 0$ and $m > 1$ be constants. Then for $0 < p < \infty$ we have*

$$f \in L^p(E) \Longleftrightarrow S = \sum_{k \geq 1} m^{kp} \left| \left\{ x \in E : f(x) > \theta m^k \right\} \right| < \infty,$$

and

$$\frac{S}{C} \leq \|f\|^p_{L^p(E)} \leq C(|E| + S),$$

where $C > 0$ is a constant depending only on θ, m, and p.

2.3 Lipschitz regularity for limiting equations

In this section, we prove interior and boundary Lipschitz regularity for limiting equations whose nonlinearities depend on the gradients of weak solutions and only one variable, say x^1, by using Campanato type embeddings.

To prove Lipschitz regularity of the solutions to limiting equations, we assume that $a(\xi, x^1) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfy the following conditions:

$$\begin{cases} a(\xi, x^1) \text{ is measurable in } x^1 \in \mathbb{R} \text{ for every } \xi \in \mathbb{R}^n, \\ a(\xi, x^1) \text{ is } C^1\text{-regular in } \xi \in \mathbb{R}^n \text{ for a.e. } x^1 \in \mathbb{R}, \end{cases} \tag{2.19}$$

and

$$\begin{cases} |a(\xi, x^1)| \leq \Lambda|\xi|, \\ |D_\xi a(\xi, x^1)| \leq \Lambda, \\ \langle D_\xi a(\xi, x^1)\zeta, \zeta\rangle \geq \lambda|\zeta|^2, \end{cases} \tag{2.20}$$

for a.e. $x^1 \in \mathbb{R}$, for every $\xi, \zeta \in \mathbb{R}^n$ and for some constants $0 < \lambda \leq \Lambda$.

2.3.1 Interior Lipschitz regularity for limiting equations

We prove interior Lipschitz regularity for the limiting equations whose nonlinearity is independent of x'-variables. Under the assumptions (2.19) and (2.20), let $w \in W^{1,2}(B_{4R})$ be a weak solution of

$$\text{div } a(Dw, x^1) = 0 \text{ in } B_{4R}. \tag{2.21}$$

We write $a_{ij}(x)$ as

$$a_{ij}(x) = \frac{\partial a^i}{\partial \xi_j}(Dw(x), x^1) \quad (x \in B_{4R}). \tag{2.22}$$

Then from (2.20), we see that $a_{ij}(x)$ satisfy

$$\begin{cases} |a_{ij}(x)| \leq \Lambda, \\ a_{ij}(x)\zeta_i\zeta_j \geq \lambda|\zeta|^2, \end{cases} \tag{2.23}$$

for a.e. $x \in B_{4R}$ and every $\zeta \in \mathbb{R}^n$.

We now return to the limiting equation (2.21). Knowing that the nonlinearity $a(\xi, x^1)$ in (2.21) is independent of x'-variables, one can linearize this problem into the form like (2.15) for the tangential derivatives $D_{x'}w$, as we now have.

Lemma 2.3.1. *Let $1 < k \leq n$. If w is a weak solution of (2.21), then we have $D_k w \in W^{1,2}(B_r(x_0))$ and the following estimates*

$$\fint_{B_\rho(x_0)} |D_k w - (D_k w)_{B_\rho(x_0)}|^2 \, dx \leq C\left(\frac{\rho}{r}\right)^{2\alpha} \fint_{B_r(x_0)} |D_k w|^2 \, dx,$$

and

$$\int_{B_{r/2}(x_0)} |DD_k w|^2 \, dx \leq \frac{C}{r^2} \int_{B_r(x_0)} |D_k w - (D_k w)_{B_r(x_0)}|^2 \, dx,$$

for any $x_0 \in B_R$, $0 < \rho < r \leq 2R$ *and for some* $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$.

Proof. Since $a(\xi, x^1)$ is independent of x^k-variable, one can use kth-difference quotient and apply the difference quotient method such as [8, Lemma 5.3], to discover that $D_k w \in W^{1,2}(B_{3R})$. We then recall (2.22) and differentiate the equation (2.21) with respect to x^k to see that

$$D_i(a_{ij}(x) D_j(D_k w)) = 0 \quad \text{in } B_{3R}.$$

We have the conclusion by applying Lemma 2.2.6 to the above equation. □

We prove that the first weak derivatives of $a^1(Dw, x^1)$ exists in $B_R(x_0)$.

Lemma 2.3.2. *If w is a weak solution of (2.21), then we have* $a^1(Dw, x^1) \in W^{1,2}(B_R(x_0))$ *with the estimate*

$$\int_{B_\rho(x_0)} |D[a^1(Dw, x^1)]|^2 \, dx \leq \frac{C}{\rho^2} \int_{B_{2\rho}(x_0)} |D_{x'} w - (D_{x'} w)_{B_{2\rho}(x_0)}|^2 \, dx,$$

for any $x_0 \in B_R$ *and* $0 < \rho \leq R$.

Proof. Fix any $1 < k \leq n$. We show that the first weak derivatives of $a^1(Dw, x^1)$ exist. Since $a^1(\xi, x^1)$ does not depend on x^k-variable, Lemma 2.3.1 implies that $D_k w \in W^{1,2}(B_R(x_0))$ and

$$D_k[a^1(Dw, x^1)] = a_{1j}(x) D_{jk} w \in L^2(B_R(x_0)). \tag{2.24}$$

On the other hand, we recall that w is a weak solution of (2.21) to find that

$$\begin{aligned} D_1[a^1(Dw, x^1)] &= -\sum_{1<i\leq n} D_i[a^i(Dw, x^1)] \\ &= -\sum_{1<i\leq n} a_{ij}(x) D_{ij} w \in L^2(B_R(x_0)). \end{aligned} \tag{2.25}$$

From (2.24), (2.25) and Lemma 2.3.1, we have $D[a^1(Dw, x^1)] \in L^2(B_R(x_0))$

with the following estimate

$$\begin{aligned}\int_{B_\rho(x_0)} |D[a^1(Dw,x^1)]|^2\,dx &\le C\int_{B_\rho(x_0)} |DD_{x'}w|^2\,dx\\ &\le \frac{C}{\rho^2}\int_{B_{2\rho}(x_0)} |D_{x'}w-(D_{x'}w)_{B_{2\rho}(x_0)}|^2\,dx.\end{aligned}$$

□

We further have the following estimate.

Lemma 2.3.3. *We denote $\hat a = a^1(Dw,x^1)$. If w is a weak solution of (2.21), then we have*

$$\fint_{B_\rho(x_0)} |\hat a-(\hat a)_{B_\rho(x_0)}|^2\,dx \le C\left(\frac{\rho}{r}\right)^{2\alpha}\fint_{B_r(x_0)} |D_{x'}w|^2dx,$$

for any $x_0\in B_R$ and $0<2\rho<r\le 2R$.

Proof. By Poincaré's inequality, we have

$$\begin{aligned}\fint_{B_\rho(x_0)} |\hat a-(\hat a)_{B_\rho(x_0)}|^2dx &\le C\rho^2\fint_{B_\rho(x_0)} |D\hat a|^2\,dx.\\ &\le C\rho^2\fint_{B_\rho(x_0)} |D[a^1(Dw,x^1)]|^2\,dx.\end{aligned}\tag{2.26}$$

We now use Lemma 2.3.1 and Lemma 2.3.2 to derive that

$$\begin{aligned}\rho^2\fint_{B_\rho(x_0)} |D[a^1(Dw,x^1)]|^2dx &\le C\int_{B_{2\rho}(x_0)} |D_{x'}w-(D_{x'}w)_{B_{2\rho}(x_0)}|^2dx\\ &\le C\left(\frac{\rho}{r}\right)^{2\alpha}\int_{B_r(x_0)} |D_{x'}w-(D_{x'}w)_{B_r(x_0)}|^2dx.\end{aligned}$$

We have the conclusion from this estimate, (2.26) and Lemma 2.2.2. □

With the constant ν_0 chosen in Lemma 2.2.5, we define

$$J(x) = (\nu_0\lambda^{-1}a^1(Dw(x),x^1), D_{x'}w(x))\quad (x\in B_{4R}).\tag{2.27}$$

We now are ready to prove interior Lipschitz regularity for (2.21).

Lemma 2.3.4. *Suppose that w is a weak solution of (2.21). Then we have $Dw \in L^\infty(B_R;\mathbb{R}^n)$ with the following estimate*

$$\|Dw\|_{L^\infty(B_R)} \leq C \left(\fint_{B_{4R}} |Dw|^2 \; dx \right)^{\frac{1}{2}}.$$

Proof. We claim that

$$\fint_{B_\rho(x_0)} |J - (J)_{B_\rho(x_0)}|^2 \; dx \leq C \left(\frac{\rho}{R}\right)^{2\alpha} \fint_{B_{2R}(x_0)} |J|^2 \; dx, \tag{2.28}$$

for any $x_0 \in B_R$, $0 < \rho \leq 2R$ and for some $\alpha = \alpha(n, \lambda, \Lambda) \in (0,1)$. If $R \leq 2\rho$, then we have $\rho \leq 2R \leq 4\rho$ and Lemma 2.2.2 implies that

$$\fint_{B_\rho(x_0)} |J - (J)_{B_\rho(x_0)}|^2 \; dx \leq C \fint_{B_\rho(x_0)} |J|^2 \; dx \leq C \left(\frac{\rho}{R}\right)^{2\alpha} \fint_{B_{2R}(x_0)} |J|^2 \; dx,$$

and (2.28) holds. If $2\rho < R$, then we have from Lemma 2.3.1 and Lemma 2.3.3 that

$$\fint_{B_\rho(x_0)} |J - (J)_{B_\rho(x_0)}|^2 \; dx \leq C \left(\frac{\rho}{R}\right)^{2\alpha} \fint_{B_{2R}(x_0)} |D_{x'} w|^2 \; dx. \tag{2.29}$$

Thus we see from (2.29) and Lemma 2.3.1 that (2.28) holds when $2\rho < R$. Thus by considering two cases: $R \leq 2\rho$ and $2\rho < R$, we see that the claim (2.28) holds.

From (2.28) and Lemma 2.2.4, we have

$$||J||_{L^\infty(B_R)} \leq C \left(\fint_{B_{4R}} |J|^2 \; dx \right)^{\frac{1}{2}}.$$

Use (2.27) and Lemma 2.2.5 when $\xi = Dw$ to find that the lemma holds. □

2.3.2 Boundary Lipschitz regularity for limiting equations

We next extend the interior Lipschitz estimate Lemma 2.3.4 to study the boundary Lipschitz estimate up to the flat boundary. To do this, we recall the assumptions (2.19) and (2.20), and then let $w \in W^{1,2}(B^+_{4R})$ be a weak solution of

$$\begin{cases} \operatorname{div} a(Dw, x^1) &= 0 \quad \text{in } B^+_{4R}, \\ w &= 0 \quad \text{on } T_{4R}. \end{cases} \tag{2.30}$$

We write $a_{ij}(x)$ as

$$a_{ij}(x) = \frac{\partial a^i}{\partial \xi_j}(Dw(x), x^1) \quad (x \in B^+_{4R}). \tag{2.31}$$

Then from (2.20), we see that $a_{ij}(x)$ satisfy

$$\begin{cases} |a_{ij}(x)| \le \Lambda, \\ a_{ij}(x)\zeta_i\zeta_j \ge \lambda|\zeta|^2, \end{cases} \tag{2.32}$$

for a.e. $x \in B^+_{4R}$ and every $\zeta \in \mathbb{R}^n$.

Next, we let

$$\begin{cases} \hat{w} \text{ be the odd extension of } w \text{ from } B^+_{4R} \text{ to } B_{4R}, \\ \hat{a} \text{ be the even extension of } a^1(Dw, x^1) \text{ from } B^+_{4R} \text{ to } B_{4R}. \end{cases} \tag{2.33}$$

Then we see that

$$D_k\hat{w} \text{ is the odd extension of } D_k w \text{ from } B^+_{4R} \text{ to } B_{4R} \text{ for } 1 < k \le n. \tag{2.34}$$

Lemma 2.3.5. *Let $1 < k \le n$. If w is a weak solution of (2.30), then we have $D_k\hat{w} \in W^{1,2}(B_{3R})$ and the following estimates*

$$\begin{cases} \displaystyle\int_{B_{\frac{r}{2}}(x_0)} |DD_k\hat{w}|^2 \, dx \le \frac{C}{r^2} \fint_{B_r(x_0)} |D_k\hat{w} - (D_k\hat{w})_{B_r(x_0)}|^2 \, dx, \\ \displaystyle\fint_{B_\rho(x_0)} |D_k\hat{w} - (D_k\hat{w})_{B_\rho(x_0)}|^2 \, dx \le C\left(\frac{\rho}{r}\right)^{2\alpha} \fint_{B_r(x_0)} |D_k\hat{w}|^2 \, dx. \end{cases}$$

for any $x_0 \in B_R$, $0 < \rho < r \le 2R$ and for some $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$.

Proof. Notice that $w = 0$ on T_{4R} in the trace sense and that $a^1(\xi, x^1)$ is independent of x'-variables. As we did in Lemma 2.3.1, one can use k-th difference quotient method such as [8, Lemma 5.3] to discover that $D_k w \in$

$W^{1,2}(B_{3R}^+)$. Then differentiate (2.30) with respect to x^k to obtain

$$\begin{cases} D_i\left[a_{ij}(x)D_j(D_k w)\right] &= 0 \quad \text{in } B_{3R}^+, \\ D_k w &= 0 \quad \text{on } T_{3R}. \end{cases} \tag{2.35}$$

We also note that $D_k w = 0$ on T_{3R} in the trace sense. By using the odd extension of $D_k w$, we extend the problem (2.35) in B_{3R}^+ to the one in B_{3R} as follows:

$$\hat{a}_{ij}(x) \text{ is an extension of } a_{ij}(x) \text{ from } B_{3R}^+ \text{ to } B_{3R}, \tag{2.36}$$

by

$$\begin{cases} \hat{a}_{11}(-x^1,x') = a_{11}(x^1,x'), & \\ \hat{a}_{i1}(-x^1,x') = -a_{i1}(x^1,x') & \text{for } 1 < i \le n, \\ \hat{a}_{1j}(-x^1,x') = -a_{1j}(x^1,x') & \text{for } 1 < j \le n, \\ \hat{a}_{ij}(-x^1,x') = a_{ij}(x^1,x') & \text{for } 1 < i \le n, 1 < j \le n, \end{cases} \tag{2.37}$$

for a.e. $(x^1,x') \in B_{3R}^+$. Then one can readily check from (2.32) that

$$\begin{cases} \|\hat{a}_{ij}\|_{L^\infty(B_{3R})} \le \Lambda, \\ \hat{a}_{ij}(x)\zeta_i\zeta_j \ge \lambda|\zeta|^2, \end{cases} \tag{2.38}$$

for a.e. $x \in B_{3R}$ and for every $\zeta \in \mathbb{R}^n$, and that $D_k\hat{w}$ is a weak solution of

$$D_i\left[\hat{a}_{ij}(x)D_j(D_k\hat{w})\right] = 0 \text{ in } B_{3R}. \tag{2.39}$$

Then by using (2.38) and (2.39), we apply Lemma 2.2.6 to (2.39) to find that

$$\fint_{B_{\frac{r}{2}}(x_0)} |DD_k\hat{w}|^2\, dx \le \frac{C}{r^2}\fint_{B_r(x_0)} |D_k\hat{w} - (D_k\hat{w})_{B_r(x_0)}|^2\, dx, \tag{2.40}$$

and

$$\fint_{B_\rho(x_0)} |D_k\hat{w} - (D_k\hat{w})_{B_\rho(x_0)}|^2\, dx \le C\left(\frac{\rho}{r}\right)^{2\alpha}\fint_{B_r(x_0)} |D_k\hat{w}|^2\, dx. \tag{2.41}$$

Thus we see from (2.40) and (2.41) that the lemma holds. $\square$

The following lemma shows that $D\hat{a}$ exists in the weak sense.

Lemma 2.3.6. *Suppose that w is a weak solution of (2.30). Then we have $D\hat{a} \in L^2(B_R(x_0))$ with the estimate*

$$\fint_{B_\rho(x_0)} |D\hat{a}|^2 \, dx \le \frac{C}{\rho^2} \sum_{1<k\le n} \fint_{B_{2\rho}(x_0)} |D_k\hat{w} - (D_k\hat{w})_{B_{2\rho}(x_0)}|^2 \, dx,$$

for any $x_0 \in B_R$ and $0 < \rho \le R$.

Proof. From (2.33) and (2.34), we have

$$\begin{cases} D_k\hat{a}(x) = a_{1j}(x) D_{jk} w(x) \text{ in } B_{3R}^+, \\ D_k\hat{a}(x) = D_k\hat{a}(-x^1, x') = a_{1j}(-x^1, x') D_{jk} w(-x^1, x') \text{ in } B_{3R}^-. \end{cases} \tag{2.42}$$

By the definition of weak derivatives, (2.30) implies that

$$D_1\hat{a}(x) = D_1[a^1(Dw, x^1)] = -\sum_{1<i\le n} D_i[a^i(Dw, x^1)] = -\sum_{1<i\le n} a_{ij}(x) D_{ij} w, \tag{2.43}$$

for any $x \in B_{3R}^+$. Then we see from (2.33) that

$$D_1\hat{a}(x) = -D_1[\hat{a}(-x^1, x')] = \sum_{1<i\le n} a_{ij}(-x^1, x') D_{ij} w(-x^1, x') \text{ in } B_{3R}^-. \tag{2.44}$$

From (2.34), we have

$$|D_{jk} w(-x^1, x')| = |D_{jk}\hat{w}(-x^1, x')| = |D_{jk}\hat{w}(x^1, x')| \quad \text{in } B_{3R}^- \tag{2.45}$$

for any $1 < k \le n$ and $1 \le j \le n$. From (2.42) - (2.45), we have

$$|D\hat{a}| \le C \sum_{1<k\le n,\, 1\le j\le n} |D_{jk}\hat{w}| \le C \sum_{1<k\le n} |DD_k\hat{w}| \quad (x \in B_{3R}),$$

which implies

$$\fint_{B_\rho(x_0)} |D\hat{a}|^2 \, dx \le C \sum_{1<k\le n} \fint_{B_\rho(x_0)} |DD_k\hat{w}|^2 dx.$$

Then we have the conclusion from Lemma 2.3.5. $\square$

From Poincaré's inequality and Lemma 2.3.6, we have the next lemma.

Lemma 2.3.7. *If w is a weak solution of (2.30), then we have*

$$\fint_{B_\rho(x_0)} \left|\hat{a} - (\hat{a})_{B_\rho(x_0)}\right|^2 \, dx \le C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{B_r(x_0)} |D_k \hat{w}|^2 \, dx,$$

for any $x_0 \in B_R$, $0 < 2\rho < r \le 2R$ and for some $\alpha = \alpha(n, \lambda, \Lambda) \in (0,1)$.

Proof. From Poincaré's inequality, we have

$$\fint_{B_\rho(x_0)} \left|\hat{a} - (\hat{a})_{B_\rho(x_0)}\right|^2 \, dx \le C\rho^2 \fint_{B_\rho(x_0)} |D\hat{a}|^2 \, dx.$$

Then from Lemma 2.3.5 and Lemma 2.3.6, we see that the lemma holds. □

With the constant ν_0 chosen in Lemma 2.2.5, we define

$$J(x) = (\nu_0 \lambda^{-1} \hat{a}(x), D_{x'} \hat{w}(x)) \quad (x \in B_{4R}). \tag{2.46}$$

We now prove the following boundary Lipschitz regularity.

Lemma 2.3.8. *Let w be a weak solution of (2.30). Then we have $Dw \in L^\infty(B_R^+; \mathbb{R}^n)$ with the following estimate*

$$\|Dw\|_{L^\infty(B_R^+)} \le C \left(\fint_{B_{4R}^+} |Dw|^2 \, dx\right)^{\frac{1}{2}}.$$

Proof. We claim that

$$\fint_{B_\rho(x_0)} |J - (J)_{B_\rho(x_0)}|^2 \, dx \le C \left(\frac{\rho}{R}\right)^{2\alpha} \fint_{B_{2R}(x_0)} |J|^2 \, dx, \tag{2.47}$$

for any $x_0 \in B_R$, $0 < \rho \le 2R$ and for some $\alpha = \alpha(n, \lambda, \Lambda) \in (0,1)$. If $R \le 2\rho$, then we have $\rho \le 2R \le 4\rho$ and

$$\fint_{B_\rho(x_0)} |J - (J)_{B_\rho(x_0)}|^2 \, dx \le C \fint_{B_\rho(x_0)} |J|^2 \, dx \le C \left(\frac{\rho}{R}\right)^{2\alpha} \fint_{B_{2R}(x_0)} |J|^2 \, dx,$$

and the claim (2.47) holds. If $2\rho < R$, then we have from Lemma 2.3.5 and Lemma 2.3.7 that (2.47) holds. Thus by considering two cases: $R \le 2\rho$ and $2\rho < R$, we see that the claim (2.47) holds.

In light of (2.47) and Lemma 2.2.4, we have

$$\|J\|_{L^\infty(B_R)} \leq C \left(\fint_{B_{4R}} |J|^2 \, dx \right)^{\frac{1}{2}} \tag{2.48}$$

From the definition of $J(x)$ in (2.46), we have

$$\|J\|_{L^\infty(B_R^+)} \leq C\|J\|_{L^\infty(B_R)} \quad \text{and} \quad \fint_{B_{4R}} |J|^2 \, dx \leq C \fint_{B_{4R}^+} |J|^2 \, dx \tag{2.49}$$

By using the definition of $D_k \hat{w}$, $\hat{a}$ and J in (2.33), (2.34) and (2.46), we have $J(x) = (\nu_0 \lambda^{-1} a^1(Dw, x^1), D_{x'} w)$ in B_{4R}^+. From (2.48) and (2.49), we have

$$\|(\nu_0 \lambda^{-1} a^1(Dw, x^1), D_{x'} w)\|^2_{L^\infty(B_R^+)} \leq C \fint_{B_{4R}^+} |(\nu_0 \lambda^{-1} a^1(Dw, x^1), D_{x'} w)|^2 dx.$$

Then we apply Lemma 2.2.5 when $\xi = Dw$ to see that the lemma holds. $\square$

2.4 Comparison estimates

We obtain comparison estimates. Comparison estimate is a standard method in the regularity theory and appears in many papers, see for instance [7, 8].

2.4.1 Interior comparison estimates

With the assumptions (2.1) and (2.2), let $u \in W^{1,2}(B_6)$ be a weak solution of

$$\operatorname{div} a(Du, x) = \operatorname{div} F \text{ in } B_6 \tag{2.50}$$

satisfying

$$\fint_{B_6} |Du|^2 \, dx \leq 1, \quad \fint_{B_6} |F|^2 \, dx \leq \delta^2 \quad \text{and} \quad \fint_{B_6} |\theta(a, \bar{a})|^2 \, dx \leq \delta^2. \tag{2.51}$$

for some $\bar{a}(\xi, x^1)$ satisfying (2.1) and (2.2), where δ is to be determined.

Let v be the weak solution of

$$\begin{cases} \operatorname{div} a(Dv, x) &= 0 \quad \text{in } B_6, \\ v &= u \quad \text{on } \partial B_6, \end{cases} \tag{2.52}$$

and w be the weak solution of

$$\begin{cases} \text{div } \bar{a}(Dw, x^1) &= 0 \quad \text{in } B_5, \\ w &= v \quad \text{on } \partial B_5. \end{cases} \tag{2.53}$$

Then the uniform bound (2.51) and the standard L^2-estimates for (2.52) and (2.53) imply

$$⨍_{B_5} |Dw|^2 \, dx \le C ⨍_{B_6} |Dv|^2 \, dx \le C ⨍_{B_6} |Du|^2 \, dx \le C. \tag{2.54}$$

We first test $u - v \in W_0^{1,2}(B_6)$ for (2.50) and (2.52), and then use (2.51) to find that

$$⨍_{B_6} |Du - Dv|^2 \, dx \le C ⨍_{B_6} |F|^2 \, dx \le C\delta^2. \tag{2.55}$$

According to a well known higher integrability result for the homogeneous problem (2.52), see for instance [24], we find that there exists a positive constant $\sigma_1 = \sigma_1(n, \lambda, \Lambda)$ such that $Dv \in L^{2+\sigma_1}(B_5; \mathbb{R}^n)$ with the estimate

$$\left(⨍_{B_5} |Dv|^{2+\sigma_1} \, dx \right)^{\frac{1}{2+\sigma_1}} \le C \left(1 + \left(⨍_{B_6} |Dv|^2 \, dx \right)^{\frac{1}{2}} \right).$$

This estimate and (2.54) imply

$$⨍_{B_5} |Dv|^{2+\sigma_1} \, dx \le C. \tag{2.56}$$

We next return the reference problem (2.53). Apply Lemma 2.3.4 and then use (2.54) to discover that there exist constants $n_1 = n_1(n, \lambda, \Lambda)$ such that

$$\|Dw\|_{L^\infty(B_4)} \le n_1. \tag{2.57}$$

We now take the test function $v - w \in W_0^{1,2}(B_5)$ for (2.52) and (2.53) to write the resulting expression as

$$A = B, \tag{2.58}$$

where

$$\begin{cases} A = \fint_{B_5} \langle \bar{a}(Dv, x^1) - \bar{a}(Dw, x^1), Dv - Dw \rangle \, dx, \\ B = \fint_{B_5} \langle \bar{a}(Dv, x^1) - a(Dv, x), Dv - Dw \rangle \, dx. \end{cases} \tag{2.59}$$

From (2.2), we have

$$\lambda \fint_{B_5} |Dv - Dw|^2 \, dx \le A. \tag{2.60}$$

From Young's inequality and (2.2), we have

$$\begin{aligned} B &\le C \fint_{B_5} |\bar{a}(Dv, x^1) - a(Dv, x)||Dv - Dw| \, dx \\ &\le C \fint_{B_5} |\theta(a, \bar{a})||Dv||Dv - Dw| \, dx \\ &\le \frac{\lambda}{2} \fint_{B_5} |Dv - Dw|^2 \, dx + C \fint_{B_5} |\theta(a, \bar{a})|^2 |Dv|^2 \, dx. \end{aligned} \tag{2.61}$$

Notice that $|\theta(a, \bar{a})| \le 2\Lambda$ is bounded. In view of Hölder's inequality, (2.51) and (2.56), we have

$$\begin{aligned} &\fint_{B_5} |\theta(a, \bar{a})|^2 |Dv|^2 \, dx \\ &\le C \left(\fint_{B_5} |\theta(a, \bar{a})|^{\frac{2(2+\sigma_1)}{\sigma_1}} \, dx \right)^{\frac{\sigma_1}{2+\sigma_1}} \left(\fint_{B_5} |Dv|^{2+\sigma_1} \, dx \right)^{\frac{2}{2+\sigma_1}} \\ &\le C\delta^{\sigma_2}, \end{aligned} \tag{2.62}$$

for some positive constant $\sigma_2 = \sigma_2(n, \lambda, \Lambda)$. We see from (2.58) to (2.62) that

$$\fint_{B_5} |Dv - Dw|^2 \, dx \le C\delta^{\sigma_2}. \tag{2.63}$$

We combine (2.55) and (2.63) to conclude

$$\fint_{B_5} |Du - Dw|^2 \, dx \le \fint_{B_5} |Du - Dv|^2 \, dx + \fint_{B_5} |Dv - Dw|^2 \, dx \le C \left(\delta^2 + \delta^{\sigma_2} \right).$$

Then we have the conclusion of this subsection from (2.57).

Lemma 2.4.1. *There exists a constant $n_1 = n_1(n, \lambda, \Lambda)$ such that the following holds. For any $\epsilon > 0$, one can find a small $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ so that for such small $\delta > 0$, if u is a weak solution of (2.50), then there exists a weak solution w of (2.53) such that*

$$\fint_{B_3} |Du - Dw|^2\, dx \leq \epsilon^2 \quad \text{and} \quad \|Dw\|_{L^\infty(B_3)} \leq n_1.$$

2.4.2 Boundary comparison estimates

We next study boundary estimates for normalized problems. To do this, we first assume

$$B_6^+ \subset \Omega_6 \subset B_6 \cap \{x^1 > -12\delta\}. \tag{2.64}$$

Under the assumptions (2.1) and (2.2), let $u \in W^{1,2}(\Omega_6)$ be a weak solution of

$$\begin{cases} \operatorname{div}\, a(Du, x) &= \operatorname{div}\, F \quad \text{in } \Omega_6, \\ u &= 0 \quad\quad\quad \text{on } \partial_w \Omega_6, \end{cases} \tag{2.65}$$

satisfying

$$\fint_{\Omega_6} |Du|^2\, dx \leq 1, \quad \fint_{\Omega_6} |F|^2\, dx \leq \delta^2 \quad \text{and} \quad \fint_{B_6} |\theta(a, \bar{a})|^2\, dx \leq \delta^2, \tag{2.66}$$

for some $\bar{a}(\xi, x^1)$ satisfying (2.1) and (2.2), where δ is to be determined.

Let $v \in W^{1,2}(\Omega_6)$ be the weak solution of

$$\begin{cases} \operatorname{div}\, a(Dv, x) &= 0 \quad \text{in } \Omega_6, \\ v &= u \quad \text{on } \partial \Omega_6, \end{cases} \tag{2.67}$$

and $w \in W^{1,2}(\Omega_5)$ be the weak solution of

$$\begin{cases} \operatorname{div}\, \bar{a}(Dw, x^1) &= 0 \quad \text{in } \Omega_5, \\ w &= v \quad \text{on } \partial_w \Omega_5. \end{cases} \tag{2.68}$$

We then consider the following reference problem

$$\begin{cases} \operatorname{div}\, \bar{a}(Dh, x^1) &= 0 \quad \text{in } B_5^+, \\ h &= 0 \quad \text{on } B_5 \cap \{x^1 = 0\}. \end{cases} \tag{2.69}$$

From (2.64), (2.66), and the standard L^2-estimates for (2.67) and (2.68), we have

$$\fint_{\Omega_5} |Dw|^2\, dx \leq C \fint_{\Omega_6} |Dv|^2\, dx \leq C \fint_{\Omega_6} |Du|^2\, dx \leq C. \tag{2.70}$$

We first compare the weak solutions of (2.65) and (2.67). Test (2.65) and (2.67) by $u - v$, and then use (2.66) to find that

$$\fint_{\Omega_6} |Du - Dv|^2\, dx \leq C \fint_{\Omega_6} |F|^2\, dx \leq C\delta^2. \tag{2.71}$$

To compare the weak solutions of (2.67) and (2.68), we need the following higher integrability result from the Reifenberg flatness condition.

Lemma 2.4.2. *There exists a positive constant $\sigma_3 = \sigma_3(n, \lambda, \Lambda)$ such that if v is a weak solution of (2.67), then we have*

$$Dv \in L^{2+\sigma_3}(\Omega_5, \mathbb{R}^n),$$

with the uniform bound

$$\fint_{\Omega_5} |Dv|^{2+\sigma_3}\, dx \leq C.$$

Proof. The Reifenberg flatness condition implies the measure density condition, which ensures that $B_6 \backslash \Omega_6$ satisfies the uniform capacity condition. Then from a well known higher integrability, see for instance [24], we have

$$\left(\fint_{\Omega_5} |Dv|^{2+\sigma_3}\, dx \right)^{\frac{1}{2+\sigma_3}} \leq C \left(1 + \left(\fint_{\Omega_6} |Dv|^2\, dx \right)^{\frac{1}{2}} \right) \leq C,$$

where we have used (2.70) for the last inequality. $\square$

We next have a comparison estimate for (2.67) and (2.68).

Lemma 2.4.3. *For any $\epsilon > 0$, one can find a small $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ so that for such small $\delta > 0$, if v is a weak solution of (2.67) and w is a a weak solution of (2.68), then we have*

$$\fint_{\Omega_5} |Dv - Dw|^2\, dx \leq \epsilon^2.$$

Proof. Test (2.67) and (2.68) by $v - w \in W_0^{1,2}(\Omega_5)$ for to find that

$$\begin{aligned} A &= \fint_{\Omega_5} \langle \bar{a}(Dv, x^1) - \bar{a}(Dw, x^1), Dv - Dw \rangle \, dx \\ &= \fint_{\Omega_5} \langle \bar{a}(Dv, x^1) - a(Dv, x), Dv - Dw \rangle \, dx = B. \end{aligned} \tag{2.72}$$

From (2.2), we estimate A as follows:

$$\lambda \fint_{\Omega_5} |Dv - Dw|^2 \, dx \leq A. \tag{2.73}$$

To estimate B, we use Young's inequality and (2.2) find that

$$\begin{aligned} B &\leq C \fint_{\Omega_5} |\bar{a}(Dv, x^1) - a(Dv, x)| \, |Dv - Dw| \, dx \\ &\leq C \fint_{\Omega_5} |\theta(a, \bar{a})||Dv||Dv - Dw| \, dx \\ &\leq \frac{\lambda}{2} \fint_{\Omega_5} |Dv - Dw|^2 \, dx + C \fint_{\Omega_5} |\theta(a, \bar{a})|^2 |Dv|^2 \, dx. \end{aligned} \tag{2.74}$$

On the other hand, (2.64), (2.66), Hölder's inequality and Lemma 2.4.2 imply

$$\begin{aligned} &\fint_{\Omega_5} |\theta(a, \bar{a})|^2 |Dv|^2 \, dx \\ &\quad \leq C \left(\fint_{\Omega_5} |\theta(a, \bar{a})|^{\frac{2(2+\sigma_3)}{\sigma_3}} \, dx \right)^{\frac{\sigma_3}{2+\sigma_3}} \left(\fint_{\Omega_5} |Dv|^{2+\sigma_3} \, dx \right)^{\frac{2}{2+\sigma_3}} \\ &\quad \leq C\delta^{\sigma_4}, \end{aligned} \tag{2.75}$$

for some positive constant $\sigma_4 = \sigma_4(n, \lambda, \Lambda)$. By combining the estimates (2.72) - (2.75), we have

$$\fint_{\Omega_5} |Dv - Dw|^2 \, dx \leq C\delta^{\sigma_4},$$

by taking $\delta > 0$ sufficiently small, we see that the lemma holds. $\square$

We now recall the Reifenberg flatness condition (2.64) to compare the weak solutions of (2.68) and (2.69) from a compactness method.

Lemma 2.4.4. *For any $\epsilon > 0$, one can find a small $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ so that for such small $\delta > 0$, if w is a weak solution of (2.68) such that*

$$\fint_{\Omega_5} |Dw|^2 \, dx \le 1, \tag{2.76}$$

then there exists a weak solution h of (2.69) such that

$$\fint_{B_5^+} |w - h|^2 \, dx \le \epsilon^2 \quad \text{and} \quad \fint_{B_5^+} |Dh|^2 \, dx \le 1.$$

Proof. We use a contradiction argument. If not, there exist $\epsilon_0 > 0$, $\{w_m\}_{m=1}^{\infty}$ and $\{\Omega_5^m\}_{m=1}^{\infty}$ such that w_m is a weak solution of

$$\begin{cases} \operatorname{div} \bar{a}(Dw, x^1) = 0 & \text{in } \Omega_5^m, \\ w_m = 0 & \text{on } \partial_w \Omega_5^m, \end{cases} \tag{2.77}$$

satisfying

$$\fint_{\Omega_5^m} |Dw_m|^2 \, dx \le 1, \tag{2.78}$$

and

$$B_5^+ \subset \Omega_5^m \subset B_5 \cap \left\{ x^1 > -\frac{12}{m} \right\}, \tag{2.79}$$

but

$$\fint_{B_5^+} |w_m - h|^2 \, dx > \epsilon_0^2 \tag{2.80}$$

for any weak solution h of

$$\begin{cases} \operatorname{div} \bar{a}(Dh, x^1) = 0 & \text{in } B_5^+, \\ h = 0 & \text{on } B_5 \cap \{x^1 = 0\}, \end{cases} \tag{2.81}$$

with

$$\fint_{B_5^+} |Dh|^2 \, dx \le 1. \tag{2.82}$$

In light of (2.78) and (2.79), we see that $\{w_m\}_{m=1}^{\infty}$ is uniformly bounded in $W^{1,2}(B_5^+)$. So there exists a subsequence of $\{w_m\}_{m=1}^{\infty}$, which we will still denote as $\{w_m\}_{m=1}^{\infty}$, and $w_0 \in W^{1,2}(B_5^+)$ such that

$$\begin{cases} Dw_m \rightharpoonup Dw_0 & \text{in } L^2(B_5^+), \\ w_m \to w_0 & \text{in } L^2(B_5^+). \end{cases} \tag{2.83}$$

From (2.77), (2.79) and (2.83), we find that $w_0 = 0$ on $B_5 \cap \{x^1 = 0\}$ in the trace sense. By using the method of *Browder and Minty*, see [8] for the details, we discover that w_0 is a weak solution of

$$\begin{cases} \operatorname{div} \bar{a}(Dw_0, x^1) & = & 0 & \text{in } B_5^+, \\ w_0 & = & 0 & \text{on } B_5 \cap \{x^1 = 0\}. \end{cases} \tag{2.84}$$

On the other hand, (2.78), (2.83) and the weak lower semi-continuity imply

$$\fint_{B_5^+} |Dw_0|^2 \, dx \leq \liminf_{m\to\infty} \fint_{\Omega_5^m} |Dw_m|^2 \leq 1. \tag{2.85}$$

Then one can reach a contradiction by taking k sufficiently large and comparing (2.80)-(2.82) to (2.83)-(2.85). This completes the proof. □

Lemma 2.4.5. *There exists a constant $n_2 = n_2(n, \lambda, \Lambda)$ such that under the assumptions and conclusions of Lemma 2.4.4, we have*

$$\fint_{\Omega_3} |Dw - D\bar{h}|^2 \, dx \leq \epsilon^2 \text{ and } \|D\bar{h}\|_{L^\infty(\Omega_3)} \leq n_2,$$

where $\bar{h}$ is the zero extension of h from B_5^+ to B_5.

Proof. By Lemma 2.4.4, we see that for any $\epsilon_1 > 0$, there exist a small $\delta_1 = \delta_1(\epsilon_1, n, \lambda, \Lambda) > 0$ and a weak solution $h \in W^{1,2}(B_5^+)$ of

$$\begin{cases} \operatorname{div} \bar{a}(Dh, x^1) & = & 0 & \text{in } B_5^+, \\ h & = & 0 & \text{on } T_5, \end{cases}$$

with

$$\fint_{B_5^+} |Dh|^2 \, dx \leq 1 \tag{2.86}$$

such that if (2.76) hold for such δ_1, then

$$\fint_{B_5^+} |w - h|^2 \, dx \leq \epsilon_1^2. \tag{2.87}$$

Then from Lemma 2.3.4 and Lemma 2.3.8 and (2.86), we have

$$\|Dh\|_{L^\infty(B_3^+)} \leq n_1 \tag{2.88}$$

for some positive constant $n_2 = n_2(n, \lambda, \Lambda)$. Let $\bar{h} \in W^{1,2}(B_5)$ be the zero extension of $h \in W^{1,2}(B_5^+)$ from B_5^+ to B_5. Then from (2.88), we have

$$\|D\bar{h}\|_{L^\infty(\Omega_3)} \leq n_2, \tag{2.89}$$

and so it only remains to show that for some small $\delta = \delta(\epsilon, n, \lambda, \Lambda) \leq \delta_1$,

$$\fint_{\Omega_3} |Dw - D\bar{h}|^2 \, dx \leq \epsilon^2. \tag{2.90}$$

A direct calculation implies that $\bar{h}$ is a weak solution of

$$\left\{ \begin{array}{rcll} \operatorname{div} \bar{a}(D\bar{h}, x^1) & = & -D_1(\bar{a}^1(Dh(0,x'),0)\chi_{\{x^1<0\}}) & \text{in } \Omega_4, \\ \bar{h} & = & 0 & \text{on } \partial_w \Omega_4. \end{array} \right. \tag{2.91}$$

Choose a cut-off function $\varphi \in C_c^\infty(B_4)$ with

$$\varphi = 1 \text{ on } B_3, \ 0 \leq \varphi \leq 1 \text{ and } |D\varphi| \leq C \text{ in } B_4. \tag{2.92}$$

Test (2.68) and (2.91) by $(w - \bar{h})\varphi^2 \in W_0^{1,2}(\Omega_4)$ to discover that

$$\begin{aligned} A &= \int_{\Omega_4} \langle \bar{a}(Dw, x^1) - \bar{a}(D\bar{h}, x^1), D[(w - \bar{h})\varphi^2] \, dx \\ &= \int_{\Omega_4 \setminus B_4^+} \bar{a}(D\bar{h}(0,x'), 0) D_1[(w - \bar{h})\varphi^2] \, dx = B. \end{aligned} \tag{2.93}$$

By Young's inequality and (2.2), we have

$$A \geq \frac{\lambda}{2} \int_{\Omega_4} |Dw - D\bar{h}|^2 \varphi^2 \, dx - C(\lambda) \int_{\Omega_4} |w - \bar{h}|^2 |D\varphi|^2 \, dx. \tag{2.94}$$

Also Hölder's inequality implies that

$$
\begin{aligned}
&\int_{\Omega_4\setminus B_4^+} |w-\bar{h}|^2|D\varphi|^2\ dx \\
&\quad = \int_{\Omega_4\setminus B_4^+} |w|^2|D\varphi|^2\ dx \\
&\quad \le \left(\int_{\Omega_4\setminus B_4^+} |w|^{\frac{2n}{n-2}}\ dx\right)^{\frac{n-2}{n}} \left(\int_{\Omega_4\setminus B_4^+} 1\ dx\right)^{\frac{2}{n}}.
\end{aligned}
\tag{2.95}
$$

From the Sobolev embedding, we find that

$$
\left(\int_{\Omega_4} |w|^{\frac{2n}{n-2}}\ dx\right)^{\frac{n-2}{n}} \le C\int_{\Omega_4} |Dw|^2 dx \le C. \tag{2.96}
$$

In view of (2.64), (2.87), (2.94), (2.95) and (2.96), we have

$$
A \ge \frac{\lambda}{2}\int_{\Omega_4} |Dw-D\bar{h}|^2\varphi^2\ dx - C\epsilon_1^2 - C\delta^{\frac{2}{n}}. \tag{2.97}
$$

On the other hand, we use (2.2), (2.70), (2.89), (2.92), Hölder's inequality and Poincaré's inequality to find that

$$
\begin{aligned}
B &\le C\int_{\Omega_4\setminus B_4^+} |\bar{a}(Dh(0,x'),0)||D_1[\varphi(w-\bar{h})]|\ dx \\
&\le C\int_{\Omega_4\setminus B_4^+} |Dh(0,x')|(|w|+|Dw|)\ dx \\
&\le C\int_{\Omega_4\setminus B_4^+} |w|+|Dw|\ dx \\
&\le C\left(\int_{\Omega_4\setminus B_4^+} |w|^2+|Dw|^2\ dx\right)^{\frac{1}{2}} \left(\int_{\Omega_4\setminus B_4^+} 1\ dx\right)^{\frac{1}{2}}.
\end{aligned}
\tag{2.98}
$$

From Poincaré's inequality and (2.74), we have

$$
\int_{\Omega_4\setminus B_4^+} |w|^2\ dx \le C\int_{\Omega_4} |w|^2\ dx \le C\int_{\Omega_4} |Dw|^2\ dx \le C. \tag{2.99}
$$

Thus from (2.98) and (2.99), we have

$$B \le C\delta^{\frac{1}{2}}. \tag{2.100}$$

We then combine (2.64), (2.92), (2.93), (2.97) and (2.100) to find that

$$\fint_{\Omega_3} |Dw - D\bar{h}|^2 \, dx \le \fint_{\Omega_4} |Dw - D\bar{h}|^2 \varphi^2 \, dx \le C\left(\epsilon_1^2 + \delta^{\frac{1}{n}} + \delta^{\frac{1}{2}}\right) \le \epsilon^2,$$

by taking ϵ_1 and the corresponding $\delta \in (0, \delta_1]$ sufficiently small. □

According to (2.71), Lemma 2.4.3 and Lemma 2.4.5, we have the following comparison estimates near the boundary.

Lemma 2.4.6. *There exists a constant $n_2 = n_2(n, \lambda, \Lambda)$ such that the following holds. For any $\epsilon > 0$, one can select a small $\delta = \delta(\epsilon, n, \lambda, \Lambda) > 0$ so that for such small $\delta > 0$, if if u is a weak solution of (2.50), then there exists a weak solution h of (2.69) such that*

$$\fint_{\Omega_3} |Du - D\bar{h}|^2 \, dx \le \epsilon^2 \quad \text{and} \quad \|D\bar{h}\|_{L^\infty(\Omega_3)} \le n_2,$$

where $\bar{h}$ is the zero extension of h from B_5^+ to B_5.

2.5 Global estimates in Reifenberg flat domains

We establish global Calderón-Zygmund estimates for elliptic problems in Reifenberg flat domains.

Lemma 2.5.1. *There exists a constant $N_1 = N_1(n, \lambda, \Lambda)$ such that the following holds. For any $\epsilon > 0$, one can select a small $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ so that for such $\delta > 0$, if $(\Omega, a(\xi, x))$ is $(\delta, 6)$-vanishing of codimension 1 and $B_8 \subset \Omega$, then for any weak solution u of (2.4) and*

$$\{x \in B_1 : \mathcal{M}(|Du|^2) \le 1\} \cap \{x \in B_1 : \mathcal{M}(|F|^2) \le \delta^2\} \ne \emptyset, \tag{2.101}$$

we have

$$|\{x \in \Omega : \mathcal{M}(|Du|^2) > N_1^2\} \cap B_1| < \epsilon |B_1|. \tag{2.102}$$

Proof. From (2.101), there exists a point $y \in B_1$ such that

$$\frac{1}{|B_\rho|}\int_{\Omega_\rho(y)} |Du|^2\,dx \le 1 \quad\text{and}\quad \frac{1}{|B_\rho|}\int_{\Omega_\rho(y)} |F|^2\,dx \le \delta^2 \quad (\rho > 0). \tag{2.103}$$

Since $B_6 \subset B_7(y) \subset B_8 \subset \Omega$, we have from (2.103) that

$$\frac{1}{|B_6|}\int_{B_6} |Du|^2\,dx \le \frac{|B_7|}{|B_6|}\frac{1}{|B_7|}\int_{B_7(y)} |Du|^2\,dx \le \left(\frac{7}{6}\right)^n. \tag{2.104}$$

Similarly, we see that

$$\frac{1}{|B_6|}\int_{B_6} |F|^2\,dx \le \left(\frac{7}{6}\right)^n \delta^2. \tag{2.105}$$

From the definition of $(\delta, 6)$-vanishing of codimension 1, there exists a coordinate system such that

$$\fint_{B_6} |\theta(a,\bar{a})|^2\,dx \le \delta^2. \tag{2.106}$$

Note that the problem (2.4) has scaling and the normalization from Lemma 2.2.1. In view of (2.104), (2.105) and (2.106), it follows from Lemma 2.4.1 that there exists a function w defined in B_4 such that

$$\fint_{B_3} |Du - Dw|^2\,dx \le \tau^2 \quad\text{and}\quad \|Dw\|_{L^\infty(B_3)} \le n_1, \tag{2.107}$$

for any $\tau > 0$ and for some $n_1 = n_1(n, \lambda, \Lambda)$.

Let $N_1^2 = \max\{4n_1^2, 3^n\}$. We claim that

$$\{x \in B_1 : \mathcal{M}(|Du|^2) > N_1^2\} \subset \{x \in B_1 : \mathcal{M}_{B_2}(|Du - Dw|^2) > n_1^2\}. \tag{2.108}$$

Recalling Definition 2.2.7, we see that

$$\begin{cases} \mathcal{M}_{B_2}(|Dw|^2) = \mathcal{M}(\chi_{B_2}|Dw|^2), \\ \mathcal{M}_{B_2}(|Du - Dw|^2) = \mathcal{M}(\chi_{B_2}|Du - Dw|^2). \end{cases} \tag{2.109}$$

Suppose $x_1 \in \{x \in B_1 : \mathcal{M}_{B_2}(|Du - Dw|^2) \leq n_1^2\}$. Then we have

$$\frac{1}{|B_\rho|} \int_{B_\rho(x_1)} \chi_{B_2} |Du - Dw|^2 \, dx \leq n_1^2 \quad (\rho > 0). \tag{2.110}$$

If $0 < \rho \leq 1$, then $B_\rho(x_1) \subset B_2$. Thus from (2.107) and (2.110), we have

$$\frac{1}{|B_\rho|} \int_{B_\rho(x_1)} |Du|^2 \, dx \leq \frac{2}{|B_\rho|} \int_{B_\rho(x_1)} \chi_{B_2} (|Du - Dw|^2 + |Dw|^2) \, dx \leq 4n_1^2.$$

If $\rho > 1$, then $B_\rho(x_1) \subset B_{2\rho} \subset B_{3\rho}(y)$. From (2.103), we have

$$\frac{1}{|B_\rho|} \int_{B_\rho(x_1)} |Du|^2 \, dx \leq \frac{3^n}{|B_{3\rho}|} \int_{B_{3\rho}(y)} |Du|^2 \, dx \leq 3^n.$$

Now the claim (2.108) follows.

We now use (2.107), (2.108) and Lemma 2.2.8 to discover

$$\begin{aligned} |\{x \in B_1 : \mathcal{M}(|Du|^2) > N_1^2\}| &\leq |\{x \in B_1 : \mathcal{M}_{B_2}(|Du - Dw|^2) > n_1^2\}| \\ &\leq \frac{C}{n_1^2} \int_{B_2} |Du - Dw|^2 \, dx \\ &\leq \frac{C\tau^2}{n_1^2} \\ &\leq \epsilon |B_1|, \end{aligned}$$

by taking a small $\tau = \tau(\epsilon, n, \lambda, \Lambda)$ and the corresponding $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ satisfying the last inequality. This completes the proof. $\square$

A scaling invariance form of the above lemma is the following.

Lemma 2.5.2. *There exists a constant $N_1 = N_1(n, \lambda, \Lambda)$ such that the following holds. For any $\epsilon > 0$, one can select a small $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ so that for such $\delta > 0$, if $(\Omega, a(\xi, x))$ is $(\delta, 6)$-vanishing of codimension 1 and $B_{8r} \subset \Omega$ with $r \in (0, 1]$, then for any weak solution u of (2.4) and*

$$\{x \in B_r : \mathcal{M}(|Du|^2) \leq 1\} \cap \{x \in B_r : \mathcal{M}(|F|^2) \leq \delta^2\} \neq \emptyset,$$

we have

$$|\{x \in \Omega : \mathcal{M}(|Du|^2) > N_1^2\} \cap B_r| < \epsilon |B_r|.$$

Proof. The proof follows from the negation of Lemma 2.5.1 and its scaling invariance property from Lemma 2.2.1. □

Lemma 2.5.3. *There exists a constant $N_2 = N_2(n, \lambda, \Lambda)$ such that the following holds. For any $\epsilon > 0$, one can select a small $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ so that for such $\delta > 0$, if $(\Omega, a(\xi, x))$ is $(\delta, 6)$-vanishing of codimension 1 and $\boldsymbol{0} \in \partial\Omega$, then for any weak solution u of (2.4) and*

$$\{x \in \Omega_1 : \mathcal{M}(|Du|^2) \leq 1\} \cap \{x \in \Omega_1 : \mathcal{M}(|F|^2) \leq \delta^2\} \neq \emptyset, \tag{2.111}$$

then we have

$$|\{x \in \Omega : \mathcal{M}(|Du|^2) > N_2^2\} \cap B_1| < \epsilon |B_1|. \tag{2.112}$$

Proof. From the definition of $(\delta, 6)$-vanishing of codimension 1, we have a coordinate system such that $\mathbf{0}$ is $-6\delta e_1$,

$$B_6^+ \subset \Omega_6 \subset B_6 \cap \{x : x^1 > -12\delta\}, \tag{2.113}$$

and

$$\fint_{B_6} |\theta(a, \bar{a})|^2 \, dxdt \leq \delta^2. \tag{2.114}$$

for some $\bar{a}(\xi, x^1)$ satisfying (2.1) and (2.2).

From (2.111), there is a point $y \in B_1(-6\delta e_1) \subset B_2$ such that

$$y \in \{x \in \Omega : \mathcal{M}(|Du|^2) \leq 1\} \cap \{x \in \Omega : \mathcal{M}(|F|^2) \leq \delta^2\} \cap B_2, \tag{2.115}$$

which implies

$$\frac{1}{|B_\rho|} \int_{\Omega_\rho(y)} |Du|^2 \, dx \leq 1 \quad \text{and} \quad \frac{1}{|B_\rho|} \int_{\Omega_\rho(y)} |F|^2 \, dx \leq \delta^2 \quad (\rho > 0). \tag{2.116}$$

Since $y \in B_2$, we have $B_6 \subset B_8(y)$. Thus we have from (2.113) that

$$\Omega_6 \subset \Omega_8(y) \quad \text{and} \quad |B_8| \leq 2 \cdot \left(\frac{4}{3}\right)^n |\Omega_6|, \tag{2.117}$$

which implies

$$\begin{cases} \dfrac{1}{|\Omega_6|}\displaystyle\int_{\Omega_6} |Du|^2\,dx \le \dfrac{|B_8|}{|\Omega_6|}\dfrac{1}{|B_8|}\displaystyle\int_{\Omega_8(y)} |Du|^2\,dx \le 2\cdot\left(\dfrac{4}{3}\right)^n, \\ \dfrac{1}{|\Omega_6|}\displaystyle\int_{\Omega_6} |F|^2\,dx \le \dfrac{|B_8|}{|\Omega_6|}\dfrac{1}{|B_8|}\displaystyle\int_{\Omega_8(y)} |F|^2\,dx \le 2\cdot\left(\dfrac{4}{3}\right)^n\delta^2. \end{cases} \tag{2.118}$$

To apply Lemma 2.4.5, we use the following normalization from Lemma 2.2.1. Set $\sigma_5^2 = 2\cdot\left(\frac{4}{3}\right)^n > 1$. Then by setting

$$\begin{cases} \tilde{u} = \sigma_5^{-1}u, & \tilde{F} = \sigma_5^{-1}F, \\ \tilde{a}(\xi,x) = \sigma_5^{-1}a(\sigma_5\xi,x), & \hat{a}(\xi,x^1) = \sigma_5^{-1}\bar{a}(\sigma_5\xi,x^1), \end{cases} \tag{2.119}$$

we find out from (2.4), (2.114), (2.118) and Lemma 2.2.1 that

$$\begin{cases} \tilde{u}_t - \operatorname{div}\tilde{a}(D\tilde{u},x,t) &=& \operatorname{div}\tilde{F} & \text{in } \Omega_6, \\ \tilde{u} &=& 0 & \text{on } \partial_w\Omega_6, \end{cases} \tag{2.120}$$

$$\begin{cases} \dfrac{1}{|\Omega_6|}\displaystyle\int_{\Omega_6} |D\tilde{u}|^2\,dx \le \dfrac{\sigma_5^{-2}}{|\Omega_6|}\displaystyle\int_{\Omega_6} |Du|^2\,dx \le 1, \\ \dfrac{1}{|\Omega_6|}\displaystyle\int_{\Omega_6} |\tilde{F}|^2\,dx \le \dfrac{\sigma_5^{-2}}{|\Omega_6|}\displaystyle\int_{\Omega_6} |F|^2\,dx \le \delta^2, \end{cases} \tag{2.121}$$

and

$$\fint_{B_6} |\theta(\tilde{a},\hat{a})|^2\,dxdt \le \delta^2. \tag{2.122}$$

By using (2.120), (2.121), (2.122) and Lemma 2.4.5, we see that for any $\tau > 0$, there exist a small $\delta = \delta(\tau,n,\lambda,\Lambda) > 0$ and $\tilde{h} \in W^{1,2}(\Omega_3)$ such that

$$\int_{\Omega_3} |D(\tilde{u}-\tilde{h})|^2\,dxdt \le \sigma_5^{-2}\tau^2 \quad \text{and} \quad \|D\tilde{h}\|_{L^\infty(\Omega_3)} \le n_2, \tag{2.123}$$

where $n_2 = n_2(n,\lambda,\Lambda)$ is a universal constant. Set $h = \sigma_5\tilde{h}$. Then we have from (2.123) that

$$\int_{\Omega_3} |D(u-h)|^2\,dxdt \le \tau^2 \quad \text{and} \quad \|Dh\|_{L^\infty(\Omega_3)} \le \sigma_5 n_2. \tag{2.124}$$

Now, we claim that

$$\{x \in B_2 : \mathcal{M}(|Du|^2) > N_2^2\} \subset \{x \in B_2 : \mathcal{M}_{\Omega_3}(|D(u-h)|^2) > n_2^2\}, \quad (2.125)$$

for $N_2^2 = \max\{4\sigma_5^2 n_2^2, 4^{n+2}\}$. To do this, suppose that

$$x_1 \in \{x \in B_2 : \mathcal{M}_{\Omega_3}(|D(u-h)|^2) \le n_2^2\}. \quad (2.126)$$

From the fact that $x_1 \in B_2$, we have

$$B_\rho(x_1) \subset B_3 \quad (0 < \rho < 1), \quad (2.127)$$

and (2.126) implies that

$$\frac{1}{|B_\rho|} \int_{\Omega_\rho(x_1)} |D(u-h)|^2 \, dxdt \le n_2^2 \quad (0 < \rho < 1). \quad (2.128)$$

If $0 < r \le 1$, then (2.124), (2.128) and the fact that $\Omega_r(x_1) \subset \Omega_3$ imply

$$\begin{aligned} \frac{1}{|B_r|} \int_{\Omega_r(x_1)} |Du|^2 dxdt &\le \frac{2}{|B_r|} \int_{\Omega_r(x_1)} |D(u-h)|^2 + |Dh|^2 dxdt \\ &\le 4\sigma_5^2 n_2^2. \end{aligned} \quad (2.129)$$

If $r > 1$, then from $x_1 \in B_{2r} \subset B_{3r}(y)$ and the fact that (2.116), we have

$$\frac{1}{|B_r|} \int_{\Omega_r(x_1)} |Du|^2 \, dxdt \le \frac{4^{n+2}}{|B_{4r}|} \int_{\Omega_{4r}(y)} |Du|^2 \, dxdt \le 4^{n+2}. \quad (2.130)$$

Thus we see from (2.129) and (2.130) that $x_1 \in \{x \in B_2 : \mathcal{M}(|Du|^2) \le N_2^2\}$ and the claim (2.125) holds.

From weak 1-1 estimate (2.2.8), (2.124) and (2.125), we finally have

$$\begin{aligned} &|\{x \in B_2 : \mathcal{M}(|Du|^2) > N_2^2\}| \\ &\quad \le |\{x \in B_2 : \mathcal{M}_{\Omega_3}(|D(u-w)|^2) > n_2^2| \\ &\quad \le \frac{C}{n_2^2} \int_{\Omega_3} |D(u-w)|^2 \, dxdt \\ &\quad \le \frac{C\tau^2}{n_2^2} \\ &\quad < \epsilon |B_1|, \end{aligned} \quad (2.131)$$

by taking $\tau = \tau(\epsilon, n, \lambda, \Lambda) > 0$ and the corresponding $\delta = \delta(\tau, n, \lambda, \Lambda) > 0$ satisfying the last inequality above. From the choice of the coordinate system, we have $-6\delta e_1 \in \partial\Omega$ and $B_1(-6\delta e_1) \subset B_2$. We transform the coordinate back so that $\mathbf{0} \in \partial\Omega$. Then we see from (2.131) that the lemma holds. $\square$

Now we denote by

$$N_0 = \max\{N_1, N_2, 1\},$$

where N_1 is given as in Lemma 2.5.2 and N_2 is given as in Lemma 2.5.3.

Lemma 2.5.4. *There exists a constant $N_0 = N_0(n, \lambda, \Lambda) \geq 1$ such that the following holds. For any $\epsilon > 0$, one can find a small $\delta = \delta(\epsilon, n, \lambda, \Lambda) > 0$ so that for such δ if $(\Omega, a(\xi, x))$ is $(\delta, 120)$-vanishing of codimension 1 and $r \in (0, 1]$, then for any weak solution u of (2.4) and*

$$|\{x \in \Omega : \mathcal{M}(|Du|^2) > N_0^2\} \cap B_r| \geq \epsilon |B_r|, \tag{2.132}$$

then

$$\Omega \cap B_r \subset \{x \in \Omega : \mathcal{M}(|Du|^2) > 1\} \cup \{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2\}. \tag{2.133}$$

Proof. We prove this lemma by contradiction. If B_r satisfies (2.132) but (2.133) is false, then there exists $y \in B_r$ such that

$$y \in \{x \in \Omega : \mathcal{M}(|Du|^2) \leq 1\} \cap \{x \in \Omega : \mathcal{M}(|F|^2) \leq \delta^2\} \cap B_r. \tag{2.134}$$

If $B_{9r} \subset \Omega$, then this contradicts Lemma 2.5.2. Thus Lemma 2.5.4 holds when $B_{9r} \subset \Omega$.

So suppose that there exists a point such that

$$x_0 \in B_{9r} \cap \partial\Omega. \tag{2.135}$$

Since $x_0 \in B_{9r}$, we have $B_r \subset B_{10r}(x_0)$ which implies

$$\begin{aligned} &|\{x \in \Omega : \mathcal{M}(|Du|^2) > N_2^2\} \cap B_r| \\ &\quad \leq |\{x \in \Omega : \mathcal{M}(|Du|^2) > N_2^2\} \cap B_{10r}(x_0)|. \end{aligned} \tag{2.136}$$

To apply Lemma 2.5.3, we use a scaling from Lemma 2.2.1 by setting

$$\begin{cases} \tilde{u}(x) = \dfrac{u(x_0+20x)}{20}, & \tilde{F}(x) = \dfrac{F(x_0+20x)}{20}, \\ \tilde{a}(\xi,x) = a(\xi, x_0+20x), & \tilde{\Omega} = \left\{ \dfrac{x-x_0}{20} : x \in \Omega \right\}. \end{cases} \tag{2.137}$$

Then from Lemma 2.2.1 and the fact that $(\Omega, a(\xi,x))$ is $(\delta,120)$-vanishing with codimension 1 for λ and Λ, we have

$$(\tilde{\Omega}, \tilde{a}(\xi,x)) \text{ is } (\delta,6)\text{-vanishing with codimension 1 for } \lambda \text{ and } \Lambda, \tag{2.138}$$

and from Lemma 2.2.1, we see that $\mathbf{0} \in \partial\tilde{\Omega}$ and $\tilde{u}$ is a weak solution of

$$\begin{cases} \tilde{u}_t - \operatorname{div} \tilde{a}(D\tilde{u},x) &= \operatorname{div} \tilde{F} & \text{in } \tilde{\Omega}, \\ \tilde{u} &= 0 & \text{on } \partial\tilde{\Omega}. \end{cases} \tag{2.139}$$

Also from (2.135) and (2.137), we have $B_{10r}(x_0) \subset B_{20r}$ and

$$\begin{aligned} &|\{x \in \Omega : \mathcal{M}(|Du|^2) > N_2^2\} \cap B_{10r}(x_0)| \\ &\quad \le 20^n |\{x \in \tilde{\Omega} : \mathcal{M}(|D\tilde{u}|^2) > N_2^2\} \cap B_r|. \end{aligned} \tag{2.140}$$

Since $y \in B_r \subset B_{10r}(x_0)$, we have $y - x_0 \in B_{10r}$ and $\frac{y-x_0}{20} \in B_r$. Thus from (2.134) and (2.137), we see that

$$\frac{y-x_0}{20} \in \{x \in \tilde{\Omega} : \mathcal{M}(|D\tilde{u}|^2) \le 1\} \cap \{x \in \tilde{\Omega} : \mathcal{M}(|\tilde{F}|^2) \le \delta^2\} \cap B_r. \tag{2.141}$$

In view of (2.138), (2.139) and (2.141), we use Lemma 2.5.3 to find that

$$|\{x \in \tilde{\Omega} : \mathcal{M}(|D\tilde{u}|^2) > N_2^2\} \cap B_r| \le \frac{\epsilon}{20^n}|B_r|. \tag{2.142}$$

Thus by using (2.138) and (2.142), we see that

$$\begin{aligned} &|\{x \in \Omega : \mathcal{M}(|Du|^2) > N_2^2\} \cap B_r| \\ &\quad \le |\{x \in \Omega : \mathcal{M}(|Du|^2) > N_2^2\} \cap B_{10r}(x_0)| \\ &\quad \le 20^n |\{x \in \tilde{\Omega} : \mathcal{M}(|D\tilde{u}|^2) > N_2^2\} \cap B_r| \\ &\quad < \epsilon |B_r|. \end{aligned} \tag{2.143}$$

Since $N_0 = \max\{N_1, N_2, 1\}$, (2.143) contradicts (2.132). This finishes the proof. $\square$

Now take ϵ, the corresponding δ and a universal constant N_0 given by Lemma 2.5.4 and set

$$\epsilon_1 = \left(\frac{10}{1-\delta}\right)^n \epsilon.$$

Then we have the following power decay estimate from Lemma 2.2.9 and Lemma 2.5.4.

Lemma 2.5.5. *Under the same assumptions in Lemma 2.5.4, we further assume*

$$|\{x \in \Omega : \mathcal{M}(|Du|^2) > N_0^2\}| < \epsilon |B_1|. \tag{2.144}$$

Then we have

$$\begin{aligned} |\{x \in \Omega : \mathcal{M}(|Du|^2) > N_0^{2k}\}| \leq & \sum_{i=1}^{k} \epsilon_1^i |\{x : \mathcal{M}(|F|^2) > \delta^2 N_0^{2(k-i)}\}| \\ & + \epsilon_1^k |\{x \in \Omega : \mathcal{M}(|Du|^2) > 1\}|. \end{aligned} \tag{2.145}$$

Proof. We prove by induction on k. The case $k = 1$ follows from Lemma 2.2.9 and Lemma 2.5.4 when

$$\begin{aligned} E &= \left\{x \in \Omega : \mathcal{M}(|Du|^2) > N_0^2\right\}, \\ F &= \left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2 \text{ or } \mathcal{M}(|Du|^2) > 1\right\}. \end{aligned}$$

Suppose that the conclusion is true for $k \geq 2$. We normalize u by $u_{N_0} = u/N_0$ and F by $F_{N_0} = F/N_0$, respectively, to see from (2.144) that

$$\begin{aligned} \left|\{x \in \Omega : \mathcal{M}(|Du_{N_0}|^2) > N_0^2\}\right| &= \left|\{x \in \Omega : \mathcal{M}(|Du|^2) > N_0^4\}\right| \\ &\leq \left|\{x \in \Omega : \mathcal{M}(|Du|^2) > N_0^2\}\right| \\ &< \epsilon |B_1|. \end{aligned}$$

Then using the induction assumption, we have

$$\begin{aligned} |\{x \in \Omega : \mathcal{M}(|Du_{N_0}|^2) > N_0^{2k}\}| \leq & \sum_{i=1}^{k} \epsilon_1^i |\{x \in \Omega : \mathcal{M}(|F_{N_0}|^2) > \delta^2 N_0^{2(k-i)}\}| \\ & + \epsilon_1^k |\{x \in \Omega : \mathcal{M}(|Du_{N_0}|^2) > 1\}|, \end{aligned}$$

and we calculate as follows:

$$\begin{aligned}|\{x\in\Omega:\mathcal{M}(|Du|^2)>N_0^{2(k+1)}\}| &= |\{x\in\Omega:\mathcal{M}(|Du_{N_0}|^2)>N_0^{2k}\}|\\ &\le \sum_{i=1}^{k}\epsilon_1^i|\{x\in\Omega:\mathcal{M}(|F_{N_0}|^2)>\delta^2N_0^{2(k-i)}\}|\\ &\quad+\epsilon_1^k|\{x\in\Omega:\mathcal{M}(|Du_{N_0}|^2)>1\}|\\ &\le \sum_{i=1}^{k+1}\epsilon_1^i|\{x\in\Omega:\mathcal{M}(|F|^2)>\delta^2N_0^{2(k+1-i)}\}|\\ &\quad+\epsilon_1^{k+1}|\{x\in\Omega:\mathcal{M}(|Du|^2)>1\}|\end{aligned}$$

as required. □

We are finally ready to prove Theorem 2.1.5.

Proof of Main Theorem. We first take $\epsilon=\epsilon(n,\lambda,\Lambda)>0$ so that

$$N_0^p 20^n\epsilon\le\frac{1}{2}, \tag{2.146}$$

and then find a corresponding $0<\delta=\delta(\epsilon,n,\lambda,\Lambda)<\frac{1}{2}$ from Lemma 2.5.5.

We next recall Lemma 2.2.1 to consider

$$u_1=\frac{\delta|\Omega|^{\frac{1}{p}}u}{\|F\|_{L^p(\Omega)}}\text{ and }F_1=\frac{\delta|\Omega|^{\frac{1}{p}}F}{\|F\|_{L^p(\Omega)}}. \tag{2.147}$$

Then by standard L^2-estimate and Hölder's inequality, we have

$$\int_\Omega|Du_1|^2dx\le C\int_\Omega|F_1|^2\le C\left(\int_\Omega|F_1|^p\right)^{\frac{2}{p}}\left(\int_\Omega 1\right)^{\frac{p-2}{p}}\le C\delta^2|\Omega|. \tag{2.148}$$

From (2.146) and weak 1-1 estimate Lemma 2.2.8, we have

$$|\{x\in\Omega:\mathcal{M}(|Du_1|^2)>N_0^2\}|\le C\int_\Omega|Du_1|^2dx\le C\delta^2|\Omega|<\epsilon|B_1|, \tag{2.149}$$

by further selecting a smaller δ such that satisfying the last inequality. We set $\epsilon_1 = \left(\frac{10}{1-\delta}\right)^n \epsilon$. Then from (2.146), we have

$$N_0^p \epsilon_1 = N_0^p \left(\frac{10}{1-\delta}\right)^n \epsilon \leq N_0^p 20^n \epsilon \leq \frac{1}{2}. \tag{2.150}$$

From Lemma 2.2.8 and (2.147), we have

$$\|\mathcal{M}(\delta^{-2}F_1^2)\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{p}{2}} \leq C\|\delta^{-2}F_1^2\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{p}{2}} \leq C|\Omega|. \tag{2.151}$$

Thus (2.151) and Lemma 2.2.10 imply that

$$\begin{aligned} &\sum_{k=i}^{\infty} N_0^{p(k-i)}|\{x \in \Omega : \mathcal{M}(|F_1|^2) > \delta^2 N_0^{2(k-i)}\}| \\ &\quad \leq C\|\mathcal{M}(\delta^{-2}F_1^2)\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{p}{2}} \\ &\quad \leq C|\Omega|. \end{aligned} \tag{2.152}$$

In view of (2.149), (2.150), (2.152), we use Lemma 2.5.5 to find that

$$\begin{aligned} &\sum_{k=1}^{\infty} N_0^{pk}|\{x \in \Omega : \mathcal{M}(|Du_1|^2) > N_0^{2k}\}| \\ &\leq \sum_{k=1}^{\infty} N_0^{pk}\left[\sum_{i=1}^{k} \epsilon_1^i|\{x : \mathcal{M}(|F_1|^2) > \delta^2 N_0^{2(k-i)}\}| + \epsilon_1^k|\{x : \mathcal{M}(|Du_1|^2) > 1\}|\right] \\ &= \sum_{i=1}^{\infty}[N_0^p\epsilon_1]^i \left[\sum_{k=i}^{\infty} N_0^{p(k-i)}|\{x \in \Omega : \mathcal{M}(|F_1|^2) > \delta^2 N_0^{2(k-i)}\}|\right] \\ &\quad + \sum_{k=1}^{\infty}\left[N_0^p\epsilon_1\right]^k |\{x \in \Omega : \mathcal{M}(|Du_1|^2) > 1\}| \\ &\leq C|\Omega| \sum_{k=1}^{\infty}\left[N_0^p\epsilon_1\right]^k \\ &\leq C|\Omega|. \end{aligned}$$

But then Lemma 2.2.8 and Lemma 2.2.10 imply

$$Du_1 \in L^p(\Omega; \mathbb{R}^n),$$

with the estimate

$$\|Du_1\|^p_{L^p(\Omega)} \leq C|\Omega|.$$

We recall the definition of u_1 and F_1 in (2.147) to conclude that

$$\|Du\|_{L^p(\Omega)} \leq C\|F\|_{L^p(\Omega)}.$$

the constant C depending only on n, λ, Λ, p and $|\Omega|$. $\square$

Chapter 3

Parabolic equations

3.1 Definitions and main result

Let $(y, s) \in \mathbb{R}^n \times \mathbb{R}$ be a typical point and let $\rho > 0$. We then start with the following notations:

1. $x = (x^1, \cdots, x^n) = (x^1, x')$ is space variables.

2. $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$, $B_\rho(y) = B_\rho + y$, $B_\rho^+ = B_\rho \cap \{x : x^1 > 0\}$.

3. $Q_\rho = B_\rho \times (-\rho^2, \rho^2)$, $Q_\rho(y, s) = Q_\rho + (y, s)$, $Q_\rho^+ = Q_\rho \cap \{x : x^1 > 0\}$, $Q_\rho^+(y, s) = Q_\rho^+ + (y, s)$, $Q_\rho^- = Q_\rho \cap \{x : x^1 < 0\}$, $Q_\rho^-(y, s) = Q_\rho^- + (y, s)$.

4. $T_\rho = Q_\rho \cap \{x^1 = 0\}$, $\Omega_\rho(y) = \Omega \cap B_r(y)$.

5. $K_\rho(y, s) = \Omega_T \cap Q_\rho(y, s)$, $\partial_w K_\rho = Q_\rho \cap [\partial\Omega \times \mathbb{R}] = [B_\rho \cap \partial\Omega] \times (-\rho^2, \rho^2)$.

6. $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$.

7. $(f)_U = \fint_U f \, dxdt$, where $f \in L^1(U)$ and $U \in \mathbb{R}^{n+1}$ is an open bounded set.

Assume $a(\xi, x, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a Carathéordory vector field, that is

$$\begin{cases} a(\xi, x, t) \text{ is measurable in } (x, t) \text{ for every } \xi \in \mathbb{R}^n, \\ a(\xi, x, t) \text{ is } C^1\text{-regular in } \xi \text{ for a.e. } (x, t) \in \mathbb{R}^n \times \mathbb{R}. \end{cases} \tag{3.1}$$

We impose the following ellipticity and growth conditions

$$\begin{cases} |a(\xi, x, t)| & \leq & \Lambda|\xi|, \\ |D_\xi a(\xi, x, t)| & \leq & \Lambda, \\ \langle D_\xi a(\xi, x, t)\zeta, \zeta\rangle & \geq & \lambda|\zeta|^2, \end{cases} \tag{3.2}$$

for a.e. $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, for every $\xi,\ \zeta \in \mathbb{R}^n$ and for some constants $0 < \lambda \leq \Lambda$.

Under the assumptions (3.1) and (3.2), we consider the following equation

$$\begin{cases} u_t - \operatorname{div}\ a(Du, x, t) & = & \operatorname{div}\ F & \text{in } \Omega_T, \\ u & = & 0 & \text{on } \partial_p\Omega_T, \end{cases} \tag{3.3}$$

where $F \in L^2(\Omega_T; \mathbb{R}^n)$ is a given vector-valued function.

As usual, we consider a weak solution in a classical parabolic Sobolev space $L^2(0, T; W_0^{1,2}(\Omega)) \cap C^0([0, T]; L^2(\Omega))$, which means that the following holds

$$\int_{\Omega_T} u\varphi_t - \langle a(Du, x, t), D\varphi\rangle\ dxdt = \int_{\Omega_T} \langle F, D\varphi\rangle\ dxdt,$$

for all $\varphi \in C_0^\infty(\Omega_T)$ with $\varphi = 0$ for $t = T$.

Remark 3.1.1. *Since weak solutions under consideration throughout this paper might not be differentiable in t-variable, we use the following Steklov average. Given a function $f \in L^1(\Omega_T)$, the Steklov average of $[f]_l$ of f is defined by*

$$[f]_l(x, t) = \begin{cases} \frac{1}{l}\int_t^{t+l} f(x, s)\ ds, & t \in (0, T - l), \\ 0 & t > T - l, \end{cases} \tag{3.4}$$

for $0 < l < T$. If u is the weak solution of (3.3), we see that for a.e. $t \in (0, T - l)$ and for all $\varphi \in W_0^{1,2}(\Omega)$, the following weak formulation

$$\int_\Omega ([u]_l)_t(x, t)\varphi(x) + \langle [a(Du, \cdot)]_l(x, t), D\varphi(x)\rangle dx = -\int_\Omega \langle [F]_l(x, t), D\varphi(x)\rangle dx,$$

holds true. Therefore, we may as well take the test function $\varphi = u$, which is possible modulo Steklov average since $u = 0$ on $\partial\Omega_T$. We refer to [19, Chapter 1] for details about the concept of the Steklov average and its application in parabolic problems.

Remark 3.1.2. *By using the same argument in [10, Remark 3], we point out that our weak solution of (3.3) will be assumed to be defined on $\Omega \times \mathbb{R}$ from the following reasons. The equation can be extended forward by taking $F = 0$ for all $t \geq T$ so that all properties in question are preserved. For backward extension, one can use the zero extension of u by taking $F = 0$ for all $t \geq 0$. Therefore from now on, there is no difference between Ω_T and $\Omega \times \mathbb{R}$.*

Definition 3.1.3. Define the Sobolev space $W^p(0, T; \Omega)$ which consists of all locally integrable function $f : \Omega_T \to \mathbb{R}$ such that

$$f \in L^p(0, T; W^{1,p}(\Omega)), f_t \in L^{p'}(0, T; W^{-1,p'}(\Omega)),$$

where $p' = \frac{p}{p-1}$ and the weak derivative in t-variable exists which defined as

$$\int_0^T f(\cdot, t)\varphi'(t)\, dt = -\int_0^T f_t(\cdot, t)\varphi(t)\, dt,$$

for all scalar functions $\varphi \in C_0^\infty(0, T)$.

Remark 3.1.4. *To use compactness argument, we introduce Aubin-Lions Lemma. We note that $W^p(0, T; \Omega)$ is a Banach space under the following norm*

$$\|f\|_{W^p(0,T)} = \|f\|_{L^p(0,T;W^{1,p}(\Omega))} + \|f_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))},$$

see [2, 21]. Also $W^p(0, T; \Omega)$ is compactly embedded in $L^p(\Omega_T)$ for $p > \frac{2n}{n+2}$, see [38, Chapter 1] for Aubin-Lions Lemma.

To measure the oscillation of $a(\xi, x, t)$ in (x', t)-variables with respect to $\bar{a}(\xi, x^1)$, we consider the next $\theta(a, \bar{a})(x, t)$ as below

$$\theta(a, \bar{a})(x, t) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\left| a(\xi, x^1, x', t) - \bar{a}(\xi, x^1) \right|}{|\xi|}, \tag{3.5}$$

for any $\bar{a}(\xi, x^1)$ satisfying the assumptions (3.1) and (3.2).

We introduce the main assumption on the nonlinearity a and the domain Ω.

Definition 3.1.5. We say $(\Omega \times \mathbb{R}, a(\xi, x, t))$ is (δ, R_0)-vanishing of codimension 1 if the following holds.

1. For every ball $B_r(x_0) \subset \Omega$ with $r \in (0, R_0]$, there exists a coordinate system depending on (x_0, t_0) and r, whose variables we still denote by $(x, t) = (x^1, x', t)$ and so that in this coordinate system (x_0, t_0) is the origin and

$$\fint_{Q_r} |\theta(a, \bar{a})(x, t)|^2 \, dxdt \leq \delta^2,$$

for some $\bar{a}(\xi, x^1)$ satisfying (3.1) and (3.2).

2. For every point $(x_0, t_0) \in \partial\Omega \times \mathbb{R}$ with $r \in (0, R_0]$, there exists a coordinate system depending on (x_0, t_0) and r, whose variables we still denote by $(x, t) = (x^1, x', t)$ such that in this new coordinate system (x_0, t_0) is $(-\delta r e_1, 0)$ and

$$B_r^+ \subset B_r \cap \Omega \subset B_r \cap \{(x^1, x') : x^1 > -2\delta r\}$$

and

$$\fint_{Q_r} |\theta(a, \bar{a})(x, t)|^2 \, dxdt \leq \delta^2,$$

for some $\bar{a}(\xi, x^1)$ satisfying (3.1) and (3.2).

Remark 3.1.6. *Throughout this section, $0 < \delta < \frac{1}{8}$ is a small constant to be determined later so that the main result Theorem 3.1.8 holds for $2 \leq p < \infty$. On the other hand, R_0 can be any number which is bigger than 1 by the scaling invariance of the problem (3.3), see Lemma 3.2.4 in the next section.*

Remark 3.1.7. *When we defined (δ, R_0)-vanishing of codimension 1 in the earlier results, we focused on the small BMO condition and the term $\bar{a}(\xi, x^1)$ in Definition 3.1.5 was specified as the mean average over (x', t)-variables by using (x_0, t_0) and r, say $\bar{a}(\xi, x^1) = \fint_{|z'-x_0'|<r, |s-t_0|<r^2} a(\xi, x^1, z', s) \, dz'ds$. So if $a(\xi, x, t)$ has a small BMO-semi norm in the category of measurable nonlinearities, then $a(\xi, x, t)$ also satisfies our new definition of (δ, R_0)-vanishing of codimension 1 in Definition 3.1.5.*

We now state the main theorem of this paper.

Theorem 3.1.8. *Suppose that $F \in L^p(\Omega_T; \mathbb{R}^n)$ for some $p \geq 2$. Then there exists $\delta = \delta(n, \lambda, \Lambda, p) > 0$ such that if $(\Omega \times \mathbb{R}, a(\xi, x, t))$ is (δ, R_0)-vanishing of codimension 1, then the unique weak solution u of (3.3) satisfies $Du \in$*

$L^p(\Omega_T;\mathbb{R}^n)$ with the following estimate

$$\int_{\Omega_T} |Du|^p \, dxdt \leq C \int_{\Omega_T} |F|^p \, dxdt,$$

where $C = C\,(n, \lambda, \Lambda, p, |\Omega|)$.

Remark 3.1.9. *From the above definition, one see that there is no regularity assumption on the nonlinearity $a(\xi, x, t)$ with respect to x^1-variable, and so there might be big jumpings of the nonlinearity $a(\xi, x, t)$ along the x^1-variable while the nonlinearity a is being averaged along the x'-variables. Note that if $(\Omega \times \mathbb{R}, a(\xi, x, t))$ is (δ, R_0)-vanishing of codimension 1, then Ω is (δ, R_0)-Reifenberg flat domain, see [7] for the concept of being δ-Reifenberg flat.*

Remark 3.1.10. *For the sake of convenience and simplicity, we employ the letter $C > 0$ throughout this paper to denote any constants which can be explicitly computed in terms of known quantities such as n, λ, Λ and p. Thus the exact value denoted by C may change from line to line in a given computation.*

3.2 Preliminaries

We start this section with the following elementary lemma.

Lemma 3.2.1. *For an open set $U \subset \mathbb{R}^{n+1}$ and $m \geq 1$, let $F \in L^2(U;\mathbb{R}^m)$. Then we have*

$$\fint_U |F - (F)_U|^2 \, dxdt \leq \fint_U |F - \xi|^2 \, dxdt,$$

for every $\xi \in \mathbb{R}^m$.

We use a parabolic version of Campanato type embedding to prove Lipschitz regularity of limiting equations in Section 3.3. The next lemma is a simple consequence of [36, Theorem 1] by using scaling.

Lemma 3.2.2. *For $m \geq 1$, let $G \in L^2(Q_{2r}, \mathbb{R}^m)$. Define*

$$[G]^2_{\mathcal{L}^{2,\kappa}(Q_r)} = \sup_{(x_0,t_0)\in Q_r,\ 0<\rho<r} \frac{1}{\rho^{2\kappa}} \fint_{Q_\rho(x_0,t_0)} |G - (G)_{Q_\rho(x_0,t_0)}|^2 \, dxdt < \infty.$$

If $0 < \kappa \le 1$, then we have

$$\|G\|_{L^\infty(Q_r)} \le C(n,\kappa)\left[\left(\fint_{Q_r} |G|^2\,dxdt\right)^{\frac{1}{2}} + r^\kappa [G]_{\mathcal{L}^{2,\kappa}(Q_r)}\right],$$

for some positive constant $C(n,\kappa)$.

The following lemma is the key for proving Lipschitz regularity.

Lemma 3.2.3. *Let $\nu_0 = \frac{\lambda}{2\Lambda} \le \frac{1}{2}$. Under the assumptions (3.1) and (3.2), we have*

$$\nu_0|\xi| \le 2|(\nu_0\lambda^{-1}a^1(\xi,x,t),\xi')| \le 4|\xi|, \tag{3.6}$$

for a.e. $(x,t)\in\mathbb{R}^n\times\mathbb{R}$ and for every $\xi\in\mathbb{R}^n$.

Proof. The second inequality is clear from (3.2). So we only prove the first inequality. From (3.2), we have

$$\lambda|\xi-\zeta|^2 \le \langle a(\xi,x,t)-a(\zeta,x,t),\xi-\zeta\rangle \le \Lambda|\xi-\zeta|^2, \tag{3.7}$$

a.e. $(x,t)\in\mathbb{R}^n\times\mathbb{R}$ and for every $\xi,\zeta\in\mathbb{R}^n$.

Fix a.e. point $(x,t)\in\mathbb{R}^n\times\mathbb{R}$ which $a(\xi,x,t)$ is C^1-regular in ξ. First, we claim that

$$|a^1(\xi_1,0',x,t)| \ge \lambda|\xi_1| \qquad (\xi\in\mathbb{R}^n). \tag{3.8}$$

If $\xi_1=0$, then (3.8) holds trivially. If $\xi_1\ne 0$, then by taking $\xi=(\xi_1,0')\in\mathbb{R}^n$ and $\zeta=(0,\cdots,0)\in\mathbb{R}^n$ in (3.7), we have

$$a^1(\xi_1,0',x,t)\xi_1 \ge \lambda|\xi_1|^2 > 0, \tag{3.9}$$

which implies (3.8). Thus we have the claim (3.8). From (3.2), we have

$$\begin{aligned}
&|a^1(\xi,x,t)-a^1(\xi_1,0',x,t)|\\
&\quad\le \left|\int_0^1 \frac{d}{ds}a^1(s\xi+(1-s)(\xi_1,0'),x,t)\,ds\right|\\
&\quad\le \int_0^1 |D_\xi a^1(s\xi+(1-s)(\xi_1,0'),x,t)||(0,\xi')|\,ds\\
&\quad\le \Lambda|\xi'|,
\end{aligned} \tag{3.10}$$

for every $\xi \in \mathbb{R}^n$. Thus it follows from (3.8) and (3.10) that

$$\begin{aligned}|a^1(\xi,x,t)| &\geq |a^1(\xi_1,0',x,t)| - |a^1(\xi,x,t) - a^1(\xi_1,0',x,t)| \\ &\geq \lambda|\xi_1| - \Lambda|\xi'|,\end{aligned} \tag{3.11}$$

for every $\xi \in \mathbb{R}^n$. Since $\nu_0 = \frac{\lambda}{2\Lambda} \leq \frac{1}{2}$, a direct calculation and (3.11) imply

$$\frac{\nu_0}{\lambda}|a^1(\xi,x,t)| + |\xi'| \geq \nu_0|\xi_1| + \left(1 - \frac{\nu_0\Lambda}{\lambda}\right)|\xi'| \geq \nu_0|\xi_1| + \frac{|\xi'|}{2} \geq \nu_0|\xi|,$$

for every $\xi \in \mathbb{R}^n$. Since $(a+b)^2 \leq 2a^2 + 2b^2$ for any $a, b \geq 0$, the above inequality implies the first inequality in (3.6). $\square$

We now recall the following properties for Steklov average. For any $g \in C([t_1, t_2 + h]; L^2(U))$ with $h > 0$, $t_1 \leq t_2$ and $U \subset \mathbb{R}^n$, we have

$$([g]_h(x,t))_t = \frac{g(x,t+h) - g(x,t)}{h} := (D_t^h g)(x,t), \tag{3.12}$$

for any $x \in U, t_1 \leq t \leq t_2$ in the weak sense. From Hölder's inequality, we have

$$\begin{aligned}\int_{U\times(t_1,t_2)} |[g]_h|^2 \, dxdt &\leq \int_{U\times(t_1,t_2)} \left|\frac{1}{h}\int_t^{t+h} g(x,\tau)\, d\tau\right|^2 \, dxdt \\ &\leq \int_{U\times(t_1,t_2)} \frac{1}{h}\int_t^{t+h} |g(x,\tau)|^2 \, d\tau \, dxdt \\ &\leq C\|g\|^2_{L^2(U\times(t_1,t_2+h))}.\end{aligned} \tag{3.13}$$

We state the scaling and normalization we are using in this section.

Lemma 3.2.4. *For each $r, M > 0$, let us define the scaling and renormalization maps:*

$$\begin{cases} \tilde{u}(x,t) = \dfrac{u(rx, r^2t)}{rM}, & \tilde{F}(x,t) = \dfrac{F(rx,r^2t)}{M}, \\ \tilde{a}(\xi,x,t) = \dfrac{a(M\xi, rx, r^2t)}{M}, & \tilde{\Omega} = \left\{\dfrac{x}{r} : x \in \Omega\right\}. \end{cases}$$

Then we have

1. *If* $u \in W_0^{1,2}(\Omega_T)$ *is a weak solution of*

$$u_t - \operatorname{div}\, a(Du, x, t) = \operatorname{div}\, F \text{ in } \Omega_T,$$

then $\tilde{u} \in W_0^{1,2}(\tilde{\Omega}_{\frac{T}{r^2}})$ *is also a weak solution of*

$$\tilde{u}_t - \operatorname{div}\, \tilde{a}(D\tilde{u}, x, t) = \operatorname{div}\, \tilde{F} \text{ in } \tilde{\Omega}_{\frac{T}{r^2}}.$$

2. *Suppose* $(\Omega \times \mathbb{R}, a(\xi, x, t))$ *is* (δ, R_0)*-vanishing with codimension 1 for constants* λ *and* Λ*. Then* $(\tilde{\Omega} \times \mathbb{R}, \tilde{a}(\xi, x, t))$ *is* $\left(\delta, \frac{R_0}{r}\right)$*-vanishing with codimension 1 for the same constants* λ *and* Λ*.*

We introduce some regularity results which will be used later in this paper. Suppose that $a_{ij}(x,t)$ satisfies

$$\begin{cases} |a_{ij}(x,t)| \le \Lambda, \\ a_{ij}(x,t)\zeta_i\zeta_j \ge \lambda|\zeta|^2, \end{cases} \tag{3.14}$$

for a.e. $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, every $\xi, \zeta \in \mathbb{R}^n$ and some positive constants $0 < \lambda \le \Lambda$.

Under the assumption (3.14), let w be a weak solution of

$$w_t - D_j[a_{ij}(x,t)D_i w] = 0 \text{ in } Q_R. \tag{3.15}$$

Then from [26, Chapter VI.12], we have the following better regularity.

Lemma 3.2.5. *If* w *be a weak solution of (3.15), then we have* $Dw \in L^2_{loc}(Q_R)$ *and the following estimates*

$$\fint_{Q_\rho} |w - (w)_{Q_\rho}|^2 \, dxdt \le C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r} |w|^2 \, dxdt,$$

and

$$\fint_{Q_\rho} |Dw|^2 \, dxdt \le \frac{C}{(r-\rho)^2} \fint_{Q_r} |w - (w)_{Q_r}|^2 \, dxdt,$$

for any $0 < \rho < r \le R$ *and for some constant* $\alpha = \alpha(n, \lambda, \Lambda) \in (0,1)$.

We use the difference quotient method, and the following variation of [21, 5.8 Theorem 3] is useful throughout this paper.

Lemma 3.2.6. *Let $\pi \in \mathbb{R}^{n+1}$ and $U \subset \mathbb{R}^{n+1}$ be an open bounded domain. Define $V_{\pi,h} \subset \mathbb{R}^{n+1}$ as*

$$V_{\pi,h} = \{(x,t) \in \mathbb{R}^{n+1} : (x,t) = (y,s) + l\pi \text{ for } (y,s) \in U \text{ and } l \in [0,h]\}.$$

(1) Let $1 \leq q < \infty$. If $D_\pi g \in L^q(V_{\pi,h})$, then we have

$$D_\pi^h g \in L^q(U) \text{ with the estimate } \|D_\pi^h g\|_{L^q(U)} \leq \|D_\pi g\|_{L^q(V_{\pi,h})}.$$

Moreover, if $D_\pi g \in L^q(V_{\pi,h_0})$ for some fixed $h_0 > 0$, then we have

$$D_\pi^h g \to D_\pi g \text{ in } L^p(U) \text{ as } h \to 0.$$

(2) Let $1 < q < \infty$. Suppose that $g \in L^q(V_{\pi,h_0})$ and $\|D_\pi^h g\|_{L^q(U)}$ is uniformly bounded in $h \in (0,h_0)$ for some fixed $h_0 > 0$. Then $D_\pi g$ exists in U with the estimate

$$\|D_\pi g\|_{L^q(U)} \leq \liminf_{h \to 0} \|D_\pi^h g\|_{L^q(U)}.$$

We will use the classical parabolic Hardy-Littlewood maximal function, the Vitali covering lemma and the standard arguments of measure theory.

Definition 3.2.7. The parabolic Hardy-Littlewood maximal function $\mathcal{M}f$ of a locally integrable function f defined in $\mathbb{R}^{n+1}$ is a function such that

$$(\mathcal{M}f)(y,s) = \sup_{\rho>0} \fint_{Q_\rho(y,s)} |f(x,t)|\, dxdt.$$

If f is defined in a bounded subset U of $\mathbb{R}^{n+1}$, we define the restricted maximal function $\mathcal{M}_U f$ by

$$\mathcal{M}_U f = \mathcal{M}\left(f\chi_U\right),$$

where χ_U is the standard characteristic function on U. We will drop the index U in $\mathcal{M}_U f$, if U is understood clearly in the context.

The basic properties for the Hardy-Littlewood maximal function are the followings.

Lemma 3.2.8. *If $f = f(x,t) \in L^p(\mathbb{R}^{n+1})$ with $1 < p \leq \infty$, then $\mathcal{M}f \in L^p(\mathbb{R}^{n+1})$ and*

$$\frac{1}{C(n,p)}\|f\|_{L^p} \leq \|\mathcal{M}f\|_{L^p} \leq C(n,p)\|f\|_{L^p}.$$

If $f \in L^1(\mathbb{R}^{n+1})$, then

$$|\{(y,s) \in \mathbb{R}^{n+1} : (\mathcal{M}f)(y,s) > \alpha\}| \leq \frac{C}{\alpha} \int |f(x,t)|\, dxdt.$$

We will use the following version of the Vitali covering lemma.

Lemma 3.2.9. *[10] Let E and F be measurable sets with $E \subset F \subset \Omega_T$. Assume that Ω is $(\delta, 1)$-Reifenberg flat. Assume further that there exists $\epsilon > 0$ such that*

$$|E| < \epsilon |Q_1|$$

and for all $(y,s) \in \Omega_T$ and for all $r \in (0,1]$ with $|F \cap Q_r(y,s)| \geq \epsilon |Q_r(y,s)|$,

$$\Omega_T \cap Q_r(y,s) \subset F.$$

Then we have

$$|E| \leq \left(\frac{10}{1-\delta}\right)^{n+2} \epsilon |F|.$$

We use the following standard arguments of measure theory.

Lemma 3.2.10. *Assume that f is a nonnegative and measurable function in $\mathbb{R}^{n+1}$. Assume further that f has a compact support in a bounded subset E of $\mathbb{R}^{n+1}$. Let $\theta > 0$ and $m > 1$ be constants. Then for $0 < p < \infty$ we have*

$$f \in L^p(E) \Longleftrightarrow S = \sum_{k \geq 1} m^{kp} \left|\left\{(x,t) \in E : f(x,t) > \theta m^k\right\}\right| < \infty$$

and

$$\frac{S}{C} \leq \|f\|^p_{L^p(E)} \leq C(|E| + S),$$

where $C > 0$ is a constant depending only on θ, m, and p.

3.3 Lipschitz regularity for limiting equations

In this section, we show Lipschitz regularity for limiting equations whose nonlinearity is independent of x'-variables. To do this, suppose that

$$\begin{cases} a(\xi, x^1) \text{ is measurable in } x^1 \in \mathbb{R} \text{ for every } \xi \in \mathbb{R}^n, \\ a(\xi, x^1) \text{ is } C^1\text{-regular in } \xi \in \mathbb{R}^n \text{ for a.e. } x^1 \in \mathbb{R}, \end{cases} \tag{3.16}$$

and $a(\xi, x^1) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfies

$$\begin{cases} |a(\xi, x^1)| \leq \Lambda |\xi|, \\ |D_\xi a(\xi, x^1)| \leq \Lambda, \\ \langle D_\xi a(\xi, x^1)\zeta, \zeta\rangle \geq \lambda |\zeta|^2, \end{cases} \tag{3.17}$$

for a.e. $x^1 \in \mathbb{R}$, for every $\xi, \zeta \in \mathbb{R}^n$ and for some constants $0 < \lambda \leq \Lambda$.

3.3.1 Interior Lipschitz regularity for limiting equations

We first prove interior Lipschitz regularity for limiting equations. Under the assumptions (3.16) and (3.17), let w be a weak solution of

$$w_t - \operatorname{div} a(Dw, x^1) \;=\; 0 \;\text{ in } Q_{4R}. \tag{3.18}$$

We write $a_{ij}(x,t) = \frac{\partial a^i}{\partial \xi_j}(Dw(x,t), x^1)$. From (3.17), we see that $a_{ij}(x,t)$ satisfies

$$\begin{cases} |a_{ij}(x,t)| \leq \Lambda, \\ a_{ij}(x,t)\zeta_i\zeta_j \geq \lambda|\zeta|^2, \end{cases} \tag{3.19}$$

for a.e. $(x,t) \in Q_{4R}$ and all $\xi, \zeta \in \mathbb{R}^n$.

Lemma 3.3.1. *Let $1 < k \leq n$. If w is a weak solution of (3.18), then we have $DD_k w \in L^2_{loc}(Q_{3R})$ with the estimates*

$$\begin{cases} \displaystyle\fint_{Q_\rho} |D_k w - (D_k w)_{Q_\rho}|^2 \, dxdt \leq C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r} |D_k w|^2 \, dxdt, \\ \displaystyle\int_{Q_\rho} |DD_k w|^2 \, dxdt \leq \frac{C}{(r-\rho)^2} \int_{Q_r} |D_k w - (D_k w)_{Q_r}|^2 \, dxdt, \end{cases}$$

for any $0 < \rho < r \leq 3R$ and for some constant $\alpha = \alpha(n, \lambda, \Lambda) \in (0,1)$.

Proof. Since $a(\xi, x^1)$ is independent of x^k-variable, one can use a difference quotient method to show that $DD_k w \in L^2(Q_{3R})$, which is possible modulo Steklov averages. Thus we differentiate (3.18) with respect to x^k-variable to obtain that $D_k w$ is a weak solution of

$$(D_k w)_t - D_i[a_{ij}(x,t) D_j(D_k w)] = 0 \text{ in } Q_{3R}, \tag{3.20}$$

where $a_{ij}(x,t)$ is defined in (3.19). By applying Lemma 3.2.5 to (3.20), we see that the lemma holds. $\square$

We show that $w_t \in L^2_{loc}(Q_{2R})$ exists in the weak sense. We use the notation $a^{t,h}_{ij}(x,t)$ in Lemma 3.3.2 and Lemma 3.3.5, which is defined as

$$a^{t,h}_{ij}(x,t) = \int_0^1 D_{\xi_j} a^i(sDw(x,t+h) + (1-s)Dw(x,t), x^1)ds, \tag{3.21}$$

for $0 < h < 3R^2$ and a.e. $(x,t) \in Q_{3R}$. Then we see from (3.17) that $a^{t,h}_{ij}(x,t)$ satisfy

$$\begin{cases} |a^{t,h}_{ij}(x,t)| \leq \Lambda, \\ a^{t,h}_{ij}(x,t)\zeta_i\zeta_j \geq \lambda|\zeta|^2, \end{cases} \tag{3.22}$$

for a.e. $(x,t) \in Q_{3R}$ and all $\zeta \in \mathbb{R}^n$. Let w be a weak solution of (3.18). Then for any $0 < h < 3R^2$ and a.e. $(x,t) \in Q_{3R}$, we have

$$\begin{aligned} &D^h_t[a^i(Dw(x,t),x^1)] \\ &\quad = \frac{a^i(Dw(x,t+h),x^1) - a^i(Dw(x,t),x^1)}{h} \\ &\quad = \frac{1}{h}\int_0^1 \frac{d}{ds} a^i(sDw(x,t+h) + (1-s)Dw(x,t), x^1)\, ds \\ &\quad = \left(\int_0^1 D_{\xi_j} a^i(sDw(x,t+h) + (1-s)Dw(x,t), x^1)ds\right) D_j(D^h_t w) \\ &\quad = a^{t,h}_{ij}(x,t) D_j(D^h_t w). \end{aligned} \tag{3.23}$$

Then we have the following lemma.

Lemma 3.3.2. *If w is a weak solution of (3.18), then we have*

$$\int_{Q_\rho} |D^h_t Dw|^2 \, dxdt \leq \frac{C}{(r-\rho)^2} \int_{Q_r} |D^h_t w|^2 \, dxdt.$$

for any $0 < \rho < r \leq 2R$ and $0 < h < R^2$.

Proof. Choose a cut-off function $\eta \in C^\infty_c(Q_r)$ such that

$$0 \leq \eta \leq 1, \eta = 1 \text{ in } Q_\rho, |D\eta|^2 \leq \frac{C}{r-\rho} \text{ and } |\eta_t| \leq \frac{C}{(r-\rho)^2}. \tag{3.24}$$

Let $0 < l < R^2$. From (3.18) and (3.23), we have

$$0 = (D_t^h w)_t - D_i[D_t^h(a^i(Dw, x^1))] = (D_t^h w)_t - D_i[a_{ij}^{t,h}(x,t) D_j(D_t^h w)] \text{ in } Q_{3R}.$$

We use Steklov average formulation, and then test by $\varphi = \eta^2 [D_t^h w]_l$ to find that

$$0 = \int_{Q_r} ([D_t^h w]_l)_t \eta^2 [D_t^h w]_l + [a_{ij}^{t,h}(x,t) D_j(D_t^h w)]_l D_i(\eta^2 [D_t^h w]_l)\, dxdt. \tag{3.25}$$

Also from the fact that $\eta \in C_c^\infty(Q_r)$, a direct calculation implies

$$\begin{aligned}\int_{Q_r} ([D_t^h w]_l)_t (\eta^2 [D_t^h w]_l)\, dxdt &= \int_{Q_r} \frac{(|[D_t^h w]_l|^2 \eta^2)_t}{2} - |[D_t^h w]_l|^2 \eta \eta_t\, dxdt \\ &= -\int_{Q_r} |[D_t^h w]_l|^2 \eta \eta_t\, dxdt.\end{aligned} \tag{3.26}$$

By sending $l \to 0$, we have from (3.25) and (3.26) that

$$0 = \int_{Q_r} |D_t^h w|^2 \eta \eta_t - a_{ij}^{t,h}(x,t) D_j(D_t^h w) D_i(\eta^2 D_t^h w)\, dxdt. \tag{3.27}$$

From (3.27), a direct calculation gives

$$\begin{aligned}&\int_{Q_r} a_{ij}^{t,h}(x,t) D_j(D_t^h w) \eta^2 D_i(D_t^h w)\, dxdt \\ &\quad = \int_{Q_r} a_{ij}^{t,h}(x,t) D_j(D_t^h w)[D_i(\eta^2 D_t^h w) - 2\eta D_i \eta D_t^h w]\, dxdt \\ &\quad = \int_{Q_r} |D_t^h w|^2 \eta \eta_t - a_{ij}^{t,h}(x,t) D_j(D_t^h w) 2\eta D_i \eta D_t^h w\, dxdt.\end{aligned} \tag{3.28}$$

We apply the ellipticity condition (3.22) and Young's inequality to (3.28), and then the choice of cut-off function η in (3.24) implies that

$$\begin{aligned}\int_{Q_r} |D_t^h Dw|^2 \eta^2\, dxdt &\le C \int_{Q_r} |D_t^h w|^2 |D\eta|^2 + |D_t^h w|^2 |\eta D_t \eta|\, dxdt \\ &\le \frac{C}{(r-\rho)^2} \int_{Q_r} |D_t^h w|^2\, dxdt.\end{aligned} \tag{3.29}$$

From (3.24) and (3.29), we see that the lemma holds. □

With the constant ν_0 in Lemma 3.2.3, we write

$$J(x,t) = (\nu_0 \lambda^{-1} a^1(Dw(x,t), x^1), D_{x'} w(x,t)) \quad ((x,t) \in Q_{4R}). \tag{3.30}$$

Then from Lemma 3.2.3, we have

$$|Dw(x,t)| \leq C|J(x,t)| \leq C|Dw(x,t)| \qquad ((x,t) \in Q_{4R}). \tag{3.31}$$

To estimate w_t in L^2, we use a variation of [17, Lemma 3.3].

Lemma 3.3.3. *If w is a weak solution of (3.18), then we have $w_t \in L^2_{loc}(Q_{2R})$ and the following estimate*

$$\int_{Q_\rho} |w_t|^2 \, dxdt \leq \frac{C}{(r-\rho)^2} \int_{Q_r} |J|^2 \, dxdt \quad (0 < \rho < r \leq 2R).$$

Proof. Let $d_0 = \dfrac{2\rho + r}{3}$, $d_\infty = \dfrac{\rho + 2r}{3}$ and $0 < h < r^2 - d_\infty^2$. For $m \geq 1$, set

$$d_m = d_0 + \sum_{l=1}^{m} \frac{r-\rho}{3 \cdot 2^l} \quad \text{and} \quad e_m = \frac{d_m + d_{m+1}}{2}.$$

Choose a cut-off function $\eta \in C_c^\infty(Q_{e_m})$ such that

$$0 \leq \eta \leq 1, \eta = 1 \text{ in } Q_{d_m}, |D\eta| \leq \frac{C2^m}{r-\rho} \text{ and } |\eta_t| \leq \frac{C4^m}{(r-\rho)^2}. \tag{3.32}$$

Use Steklov average formulation on (3.18) and then test by $\eta^2 D_t^h w$ to find that

$$\int_{Q_{e_m}} ([w]_h)_t \eta^2 (D_t^h w) \, dxdt = -\int_{Q_{e_m}} [a^i(Dw, x^1)]_h D_i[\eta^2 D_t^h w] \, dxdt. \tag{3.33}$$

From (3.12), we have $([w]_h)_t = D_t^h w$. By using this equality and Young's inequality, we see from (3.33) that

$$\begin{aligned}\int_{Q_{e_m}} \eta^2 |D_t^h w|^2 \, dxdt \le \int_{Q_{e_m}} \kappa |D_t^h Dw|^2 \eta^2 + \frac{\eta^2 |D_t^h w|^2}{4} \, dxdt \\ + C \int_{Q_{e_m}} \left[\frac{\eta^2}{\kappa} + |D\eta|^2 \right] |[a^i(Dw, x^1)]_h|^2 \, dxdt.\end{aligned} \tag{3.34}$$

Now, take $\rho = e_m$ and $r = d_{m+1}$ in Lemma 3.3.2 to find that

$$\int_{Q_{e_m}} |D_t^h Dw|^2 \, dxdt \le \frac{C_1 4^m}{(r-\rho)^2} \int_{Q_{d_{m+1}}} |D_t^h w|^2 \, dxdt. \tag{3.35}$$

Take κ so that $\dfrac{C_1 \kappa 4^m}{(r-\rho)^2} = \dfrac{1}{18}$. Then by combining (3.32), (3.34) and (3.35), we have from the ellipticity condition (3.17) that

$$\int_{Q_{d_m}} |D_t^h w|^2 \, dxdt \le \int_{Q_{d_{m+1}}} \frac{|D_t^h w|^2}{9} + \frac{C 4^m |[a^i(Dw, x^1)]_h|^2}{(r-\rho)^2} \, dxdt. \tag{3.36}$$

Multiply (3.36) by $\dfrac{1}{9^m}$ and sum it from $m = 0$ to ∞. Then we see that

$$\begin{aligned}&\sum_{m=0}^{\infty} \frac{1}{9^m} \int_{Q_{d_m}} |D_t^h w|^2 \, dxdt \\ &\quad \le \sum_{m=0}^{\infty} \frac{1}{9^{m+1}} \int_{Q_{d_{m+1}}} |D_t^h w|^2 \, dxdt \\ &\qquad + \sum_{m=0}^{\infty} \left(\frac{4}{9}\right)^m \frac{C}{(r-\rho)^2} \int_{Q_{d_{m+1}}} |[a^i(Dw, x^1)]_h|^2 \, dxdt.\end{aligned} \tag{3.37}$$

Since $d_\infty = \dfrac{\rho + 2r}{3}$ and $0 < h < r^2 - d_\infty^2$, we have from (3.13) and (3.37) that

$$\begin{aligned}\int_{Q_{d_0}} |D_t^h w|^2 \, dxdt &\le \frac{C}{(r-\rho)^2} \int_{Q_{d_\infty}} |[a^i(Dw, x^1)]_h|^2 \, dxdt \\ &\le \frac{C}{(r-\rho)^2} \int_{Q_r} |a^i(Dw, x^1)|^2 \, dxdt.\end{aligned} \tag{3.38}$$

Since $d_0 = \dfrac{2\rho + r}{3}$, we have from (3.31) and (3.38) that

$$\begin{aligned}\int_{Q_{\frac{2\rho+r}{3}}} |D_t^h w|^2 \, dxdt &\le \frac{C}{(r-\rho)^2} \int_{Q_r} |Dw|^2 \, dxdt \\ &\le \frac{C}{(r-\rho)^2} \int_{Q_r} |J|^2 \, dxdt.\end{aligned} \tag{3.39}$$

Since $0 < h < r^2 - d_\infty^2$ was arbitrary chosen, we find from (3.39) and Lemma 3.2.6 that $w_t \in L^2(Q_\rho)$ with the estimate

$$\int_{Q_\rho} |w_t|^2 \, dxdt \le \frac{C}{(r-\rho)^2} \int_{Q_r} |J|^2 \, dxdt. \tag{3.40}$$

□

Lemma 3.3.4. *If w is a weak solution of (3.18), then we have $Dw_t \in L^2_{loc}(Q_{2R})$ with the estimates*

$$\int_{Q_\rho} |Dw_t|^2 \, dxdt \le \frac{C}{(r-\rho)^2} \int_{Q_r} |w_t|^2 \, dxdt \quad (0 < \rho < r < 2R).$$

Proof. From Lemma 3.3.3, we have $w_t \in L^2(Q_r)$. Then Lemma 3.3.2 implies

$$\int_{Q_{\frac{2\rho+r}{3}}} |D_t^h Dw|^2 dxdt \le \frac{C}{(r-\rho)^2} \int_{Q_{\frac{\rho+2r}{3}}} |D_t^h w|^2 dxdt \le \frac{C}{(r-\rho)^2} \int_{Q_r} |w_t|^2 dxdt,$$

for any $0 < h < r^2 - \left(\frac{\rho+2r}{3}\right)^2$. Then we have the conclusion from Lemma 3.2.6. □

Lemma 3.3.5. *If w is a weak solution of (3.18), then we have*

$$⨍_{Q_\rho} |w_t - (w_t)_{Q_\rho}|^2 \, dxdt \le C \left(\frac{\rho}{r}\right)^{2\alpha} ⨍_{Q_r} |w_t|^2 \, dxdt,$$

for any $0 < \rho \le r < 2R$ and for some constant $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$.

Proof. If $r \le 4\rho$, then we have from Lemma 3.2.1 that

$$⨍_{Q_\rho} |w_t - (w_t)_{Q_\rho}|^2 \, dxdt \le C ⨍_{Q_\rho} |w_t|^2 \, dxdt \le C \left(\frac{\rho}{r}\right)^{2\alpha} ⨍_{Q_r} |w_t|^2 \, dxdt,$$

which implies the lemma. So suppose that $4\rho < r$.

Let $0 < h < \frac{r^2}{4}$. From (3.18) and (3.23), we have

$$\begin{aligned}0 &= (D_t^h w)_t - D_i[D_t^h(a^i(Dw, x^1))] \\ &= (D_t^h w)_t - D_i[a_{ij}^{t,h}(x,t)D_j(D_t^h w)] \quad \text{in } Q_{3R}.\end{aligned}$$

Thus we see that $D_t^h w$ is a weak solution of

$$(D_t^h w)_t - D_i[a_{ij}^{t,h}(x,t)D_j(D_t^h w)] = 0 \quad \text{in } Q_{3R}. \tag{3.41}$$

By using (3.22) and applying Lemma 3.2.5 to (3.41), we have

$$⨍_{Q_\rho} |D_t^h w - (D_t^h w)_{Q_\rho}|^2 \, dxdt \le C \left(\frac{\rho}{r}\right)^{2\alpha} ⨍_{Q_{\frac{r}{2}}} |D_t^h w|^2 \, dxdt. \tag{3.42}$$

In view of Lemma 3.3.3 and Lemma 3.2.6, we find that

$$⨍_{Q_{\frac{r}{2}}} |D_t^h w|^2 \, dxdt \le C ⨍_{Q_r} |w_t|^2 \, dxdt \tag{3.43}$$

From Lemma 3.3.3, we have $w_t \in L^2_{loc}(Q_{2R})$. Thus from Lemma 3.2.6, we have

$$D_t^h w \to D_t w \text{ in } L^2_{loc}(Q_{2R}) \text{ as } h \to 0,$$

which implies

$$⨍_{Q_\rho} |w_t - (w_t)_{Q_\rho}|^2 \, dxdt \le \liminf_{h\to 0} ⨍_{Q_\rho} |D_t^h w - (D_t^h w)_{Q_\rho}|^2 \, dxdt. \tag{3.44}$$

Thus we see from (3.42), (3.43) and (3.44) that the lemma holds. □

Since the estimate in Lemma 3.3.5 is invariant under translation and scaling, Lemma 3.2.2 implies the following lemma.

Lemma 3.3.6. *Let $0 < r \le R$. If w is a weak solution of (3.18), then we have*

$$\|w_t\|_{L^\infty(Q_{\frac{r}{4}})} \le C \left(⨍_{Q_{\frac{r}{2}}} |w_t|^2 \, dxdt \right)^{\frac{1}{2}}.$$

Proof. Choose any point $(x_0, t_0) \in Q_{\frac{r}{4}}$. Since Lemma 3.3.5 is invariant under translation and scaling, we have

$$\begin{aligned} \frac{1}{\rho^{2\alpha}} \fint_{Q_\rho(x_0,t_0)} |w_t - (w_t)_{Q_\rho(x_0,t_0)}|^2 \, dxdt &\le \frac{C}{r^{2\alpha}} \fint_{Q_{\frac{r}{4}}(x_0,t_0)} |w_t|^2 \, dxdt \\ &\le \frac{C}{r^{2\alpha}} \fint_{Q_{\frac{r}{2}}} |w_t|^2 \, dxdt, \end{aligned} \tag{3.45}$$

for any $0 < \rho \le \frac{r}{4}$ and for some constant $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$. Thus from (3.45) and Lemma 3.2.2, we have

$$\|w_t\|_{L^\infty(Q_{\frac{r}{4}})} \le C \left(\fint_{Q_{\frac{r}{2}}} |w_t|^2 \, dxdt \right)^{\frac{1}{2}}. \tag{3.46}$$

□

Lemma 3.3.7. *We denote $\hat{a} = a^1(Dw, x^1)$. If w is a weak solution of (3.18), then we have*

$$\fint_{Q_\rho} \left| \hat{a} - (\hat{a})_{Q_\rho} \right|^2 \, dxdt \le C \left(\frac{\rho}{r} \right)^{2\alpha} \fint_{Q_r} |J|^2 \, dxdt,$$

for any $0 < 8\rho \le r \le R$ and for some constant $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$.

Proof. We show that $D[a^1(Dw(x,t), x^1)]$ exists in the weak sense. Since $a^1(\xi, x^1)$ is independent of x^k-variable for $1 < k \le n$, Lemma 3.3.1 implies that

$$D_k \left[a^1(Dw(x,t), x^1) \right] = a_{1j}(x,t) D_{jk} w \in L^2(Q_r) \qquad (1 < k \le n). \tag{3.47}$$

Since w is a weak solution of (3.18), we have from Lemma 3.3.1 and Lemma 3.3.3 that $DD_{x'} w \in L^2(Q_r)$, $w_t \in L^2(Q_r)$ and

$$\begin{aligned} \int_{Q_r} [a^1(Dw(x,t), x^1)] D_1 \eta \, dxdt &= \int_{Q_r} w \eta_t - \sum_{1<i\le n} a^i(Dw(x,t), x^1) D_i \eta \, dxdt \\ &= -\int_{Q_r} w_t \eta + \sum_{1<i\le n} a_{ij}(x,t)(D_{ij} w) \eta \, dxdt, \end{aligned}$$

for any $\eta \in C_c^\infty(Q_r)$. By the definition of weak derivatives, we have

$$D_1\left[a^1(Dw(x,t),x^1)\right] = w_t - \sum_{1<i\leq n} a_{ij}(x,t)D_{ij}w \in L^2(Q_r). \tag{3.48}$$

Then we use Poincaré's inequality and a scaling argument to find that

$$\begin{aligned}
&\fint_{Q_\rho} \left|\hat{a} - (\hat{a})_{Q_\rho}\right|^2 \, dxdt \\
&\quad\leq C\fint_{Q_\rho} \rho^2 \left|D\hat{a}\right|^2 + \rho^4 \left|\hat{a}_t\right|^2 \, dxdt \\
&\quad\leq C\fint_{Q_\rho} \rho^2 \left|D[a^1(Dw,x^1)]\right|^2 + \rho^4 \left|(a^1(Dw,x^1))_t\right|^2 \, dxdt.
\end{aligned} \tag{3.49}$$

We now estimate the right-hand side of (3.49). From Lemma 3.3.6, we have

$$\fint_{Q_{2\rho}} w_t^2 \, dxdt \leq \|w_t\|^2_{L^\infty(Q_{2\rho})} \leq \|w_t\|^2_{L^\infty(Q_{\frac{r}{4}})} \leq C\fint_{Q_{\frac{r}{2}}} w_t^2 \, dxdt. \tag{3.50}$$

We use (3.47) and (3.48), and then apply Lemma 3.3.1 to find that

$$\begin{aligned}
&\fint_{Q_\rho} \left|D[a^1(Dw(x,t),x^1)]\right|^2 \, dxdt \\
&\quad\leq C\fint_{Q_\rho} w_t^2 \, dxdt + C\sum_{1<k\leq n}\fint_{Q_\rho} |DD_kw|^2 \, dxdt \\
&\quad\leq C\fint_{Q_{2\rho}} w_t^2 \, dxdt + \frac{C}{\rho^2}\sum_{1<k\leq n}\fint_{Q_{2\rho}} |D_kw - (D_kw)_{Q_{2\rho}}|^2 \, dxdt.
\end{aligned} \tag{3.51}$$

By taking $\rho = \frac{r}{2}$ in Lemma 3.3.3, we have fom (3.50) that

$$\rho^2\fint_{Q_{2\rho}} w_t^2 \, dxdt \leq C\rho^2\fint_{Q_{\frac{r}{2}}} w_t^2 \, dxdt \leq \frac{C\rho^2}{r^2}\fint_{Q_r} |J|^2 \, dxdt. \tag{3.52}$$

On the other hand, Lemma 3.3.1 and (3.31) imply that

$$\begin{aligned} \fint_{Q_{2\rho}} |D_k w - (D_k w)_{Q_{2\rho}}|^2 \, dxdt &\leq C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r} |D_k w|^2 \, dxdt \\ &\leq C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r} |J|^2 \, dxdt. \end{aligned} \tag{3.53}$$

for any $1 < k \leq n$. Since $\alpha \in (0,1)$, we have from (3.51), (3.52) and (3.53) that

$$\fint_{Q_\rho} \rho^2 |D[a^1(Dw, x^1)]|^2 \, dxdt \leq C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r} |J|^2 \, dxdt. \tag{3.54}$$

By using (3.17) and Lemma 3.3.4, we have

$$\begin{aligned} \fint_{Q_\rho} |(a^1(Dw(x,t), x^1))_t|^2 \, dxdt &\leq C \fint_{Q_\rho} |Dw_t|^2 \, dxdt \\ &\leq \frac{C}{\rho^2} \fint_{Q_{2\rho}} w_t^2 \, dxdt. \end{aligned} \tag{3.55}$$

Since $\alpha \in (0,1)$, we have from (3.52), (3.54) and (3.55) that

$$\begin{aligned} &\fint_{Q_\rho} \rho^2 |D[a^1(Dw, x^1)]|^2 + \rho^4 |(a^1(Dw, x^1))_t|^2 dxdt \\ &\quad \leq C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r} |J|^2 dxdt. \end{aligned} \tag{3.56}$$

By combining (3.49) and (3.56), we see that the lemma holds. $\square$

Lemma 3.3.8. *If w is a weak solution of (3.18), then we have*

$$\fint_{Q_\rho} \left| J - (J)_{Q_\rho} \right|^2 dxdt \leq C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r} |J|^2 dxdt,$$

for any $0 < \rho \leq r \leq 4R$ and for some constant $\alpha = \alpha(n, \lambda, \Lambda) \in (0,1)$.

Proof. Recall the definition of $J(x,t)$ in (3.30). We prove the lemma by considering four cases: $r \leq 8\rho$, $8\rho < r \leq R$, $8\rho < R < r$ and $R \leq 8\rho < r$.

From Lemma 3.2.1, we have

$$\fint_{Q_\rho} \left|J - (J)_{Q_\rho}\right|^2 \; dxdt \leq \fint_{Q_\rho} |J|^2 \; dxdt \leq C \fint_{Q_r} |J|^2 \; dxdt \quad (r \leq 8\rho). \tag{3.57}$$

From Lemma 3.3.1 and Lemma 3.3.7, we have

$$\fint_{Q_\rho} \left|J - (J)_{Q_\rho}\right|^2 \; dxdt \leq C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r} |J|^2 \, dxdt \quad (8\rho < r \leq R). \tag{3.58}$$

If $R < r$, then we have $R < r \leq 4R$. Thus Lemma 3.3.1 and Lemma 3.3.7 imply that

$$\begin{aligned} \fint_{Q_\rho} \left|J - (J)_{Q_\rho}\right|^2 dxdt &\leq C \left(\frac{\rho}{R}\right)^{2\alpha} \fint_{Q_R} |J|^2 \, dxdt \\ &\leq C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r} |J|^2 \, dxdt \qquad (8\rho < R < r). \end{aligned} \tag{3.59}$$

If $R \leq 8\rho < r$, then we have $R \leq 8\rho < r \leq 4R$ which implies $r \leq 32\rho$. Thus Lemma 3.2.1 implies that

$$\fint_{Q_\rho} \left|J - (J)_{Q_\rho}\right|^2 \; dxdt \leq \fint_{Q_\rho} |J|^2 \; dxdt \leq C \fint_{Q_r} |J|^2 \; dxdt \quad (R \leq 8\rho < r). \tag{3.60}$$

From (3.57), (3.58), (3.59) and (3.60), we see that the lemma holds. $\square$

Now, we prove Lipschitz regularity of w.

Lemma 3.3.9. *If w is a weak solution of (3.18), then w is Lipschitz continuous in Q_R with the estimate*

$$\|Dw\|_{L^\infty(Q_R)} \leq C\|Dw\|_{L^2(Q_{4R})}.$$

Proof. Since Lemma 3.3.8 is invariant under translation and scaling, we have from Lemma 3.3.8 that

$$\begin{aligned} \frac{1}{\rho^{2\alpha}} \fint_{Q_\rho(x_0,t_0)} \left|J - (J)_{Q_\rho(x_0,t_0)}\right|^2 dxdt &\leq \frac{C}{R^{2\alpha}} \fint_{Q_R(x_0,t_0)} |J|^2 \, dxdt \\ &\leq \frac{C}{R^{2\alpha}} \fint_{Q_{4R}} |J|^2 \, dxdt, \end{aligned} \tag{3.61}$$

for any $(x_0, t_0) \in Q_R$ and $0 < \rho \leq R$. Thus from (3.61) and Lemma 3.2.2, we have

$$\|J\|_{L^\infty(Q_R)} \leq C \left[\fint_{Q_R} |J|^2 dxdt \right]^{\frac{1}{2}} + CR^\alpha [J]_{L^{2,\alpha}(Q_R)} \leq C \left[\fint_{Q_{4R}} |J|^2 dxdt \right]^{\frac{1}{2}}.$$

Then from (3.31), we see that the lemma holds. $\square$

3.3.2 Boundary Lipschitz regularity for limiting equations

We extend the interior Lipschitz regularity obtained in Subsection 3.3.1 to the boundary Lipschitz regularity. The argument is parallel to the interior case, but we use odd and even extensions to obtain Lipschitz regularity of weak solutions to limiting equations.

Under the assumptions (3.16) and (3.17), let w be a weak solution of

$$\begin{cases} w_t - \operatorname{div} a(Dw, x^1) &= 0 \text{ in } Q_{4R}^+, \\ w &= 0 \text{ on } T_{4R}. \end{cases} \tag{3.62}$$

Set $a_{ij}(x,t) = \frac{\partial a^i}{\partial \xi_j}(Dw(x,t), x^1)$. Then we have

$$\begin{cases} |a_{ij}(x,t)| \leq \Lambda, \\ a_{ij}(x,t)\zeta_i\zeta_j \geq \lambda|\zeta|^2, \end{cases} \tag{3.63}$$

for a.e. $(x,t) \in Q_{4R}^+$ and every $\xi, \zeta \in \mathbb{R}^n$.

Let $1 < k \leq n$. Recall that $w = 0$ on T_{4R}. We let

$$\hat{w} \text{ be the odd extension of } w \text{ from } Q_{4R}^+ \text{ to } Q_{4R}. \tag{3.64}$$

Then we see that

$$D_k\hat{w} \text{ is the odd extension of } D_k w \text{ from } Q_{4R}^+ \text{ to } Q_{4R}. \tag{3.65}$$

Since $a(\xi, x^1)$ does not depend on x^k-variable and $w = 0$ on T_{4R}, one can use a difference quotient method to show $DD_k w \in L^2(Q_{3R}^+)$, which is possible modulo Steklov averages. Thus we differentiate (3.62) with respect to x^k-

variable to find that

$$\begin{cases} (D_k w)_t - D_i[a_{ij}(x,t)D_j(D_k w)] &= 0 \text{ in } Q^+_{3R}, \\ D_k w &= 0 \text{ on } T_{3R}. \end{cases} \tag{3.66}$$

Also let $\hat{a}_{ij}(x,t)$ be an extensions of $a_{ij}(x,t)$ from Q^+_{3R} to Q_{3R} such that

$$\begin{cases} \hat{a}_{11}(-x^1,x',t) = a_{11}(x^1,x',t), \\ \hat{a}_{1j}(-x^1,x',t) = -a_{1j}(x^1,x',t) & \text{for } 1 < j \le n, \\ \hat{a}_{i1}(-x^1,x',t) = -a_{i1}(x^1,x',t) & \text{for } 1 < i \le n, \\ \hat{a}_{ij}(-x^1,x',t) = a_{ij}(x^1,x',t) & \text{for } 1 < i \le n, 1 < j \le n, \end{cases} \tag{3.67}$$

for $(x^1,x') \in Q^+_{3R}$. One can directly check that $\hat{a}_{ij}(x,t)$ satisfy

$$\begin{cases} \|\hat{a}_{ij}(x,t)\|_{L^\infty(Q_{3R})} \le \Lambda, \\ \hat{a}_{ij}(x,t)\zeta_i\zeta_j \ge \lambda|\zeta|^2, \end{cases} \tag{3.68}$$

for a.e. $(x,t) \in Q_{3R}$ and every $\xi, \zeta \in \mathbb{R}^n$. Since $D_k\hat{w} = 0$ on T_{3R}, one can directly check from (3.65), (3.66) and (3.67) that $D_k\hat{w}$ is a weak solution of

$$(D_k\hat{w})_t - D_i[\hat{a}_{ij}(x,t)D_j(D_k\hat{w})] \;=\; 0 \text{ in } Q_{3R}. \tag{3.69}$$

Lemma 3.3.10. *Let $1 < k \le n$ and $(x_0,t_0) \in Q_R$. If w is a weak solution of (3.62), then we have $DD_k\hat{w} \in L^2_{loc}(Q_{2R}(x_0,t_0))$ with the following estimates*

$$\begin{cases} \displaystyle ⨍_{Q_\rho(x_0,t_0)} |D_k\hat{w} - (D_k\hat{w})_{Q_\rho}|^2\, dxdt \le C\left(\frac{\rho}{r}\right)^{2\alpha} ⨍_{Q_r(x_0,t_0)} |D_k\hat{w}|^2\, dxdt, \\ \displaystyle \int_{Q_\rho(x_0,t_0)} |DD_k\hat{w}|^2\, dxdt \le \frac{C}{(r-\rho)^2} \int_{Q_r(x_0,t_0)} |D_k\hat{w} - (D_k\hat{w})_{Q_r}|^2\, dxdt, \end{cases}$$

for any $0 < \rho < r \le 2R$ and for some constant $\alpha = \alpha(n,\lambda,\Lambda) \in (0,1)$.

Proof. By using (3.68), apply Lemma 3.2.5 to (3.69), Then we find that the lemma holds. ∎

We use the following notation $a^{t,h}_{ij}(x,t)$ in Lemma 3.3.11 and Lemma 3.3.14

$$a^{t,h}_{ij}(x,t) = \int_0^1 D_{\xi_j} a^i(sDw(x,t+h) + (1-s)Dw(x,t), x^1)ds, \tag{3.70}$$

for $0 < h < 3R^2$ and a.e. $(x,t) \in Q^+_{3R}$. From (3.17), we find that

$$\begin{cases} |a^{t,h}_{ij}(x,t)| \le \Lambda, \\ a^{t,h}_{ij}(x,t)\zeta_i\zeta_j \ge \lambda|\zeta|^2, \end{cases} \tag{3.71}$$

for a.e. $(x,t) \in Q^+_{3R}$ and for every $\zeta \in \mathbb{R}^n$. Then we have

$$\begin{aligned} &D^h_t[a^i(Dw(x,t),x^1)] \\ &\quad= \frac{a^i(Dw(x,t+h),x^1) - a^i(Dw(x,t),x^1)}{h} \\ &\quad= \frac{1}{h}\int_0^1 \frac{d}{ds}a^i(sDw(x,t+h)+(1-s)Dw(x,t),x^1)\,ds \\ &\quad= \left(\int_0^1 D_{\xi_j}a^i(sDw(x,t+h)+(1-s)Dw(x,t),x^1)ds\right) D_j(D^h_t w) \\ &\quad= a^{t,h}_{ij}(x,t)D_j(D^h_t w), \end{aligned} \tag{3.72}$$

for any $0 < h < 3R^2$ and a.e. $(x,t) \in Q^+_{3R}$. From (3.62) and (3.72), we have

$$\begin{aligned} 0 &= (D^h_t w)_t - D_i[D^h_t(a(Dw,x^1))] \\ &= (D^h_t w)_t - D_i[a^{t,h}_{ij}(x,t)D_j(D^h_t w)] \text{ in } Q^+_{3R}. \end{aligned} \tag{3.73}$$

In view of (3.64), we have $D^h_t\hat{w} = D^h_t w$ in Q^+_{3R}. Thus from the next calculation

$$\begin{aligned} D^h_t\hat{w}(-x^1,x',t) &= \frac{\hat{w}(-x^1,x',t+h) - \hat{w}(-x^1,x',t)}{h} \\ &= -\frac{\hat{w}(x^1,x',t+h) - \hat{w}(x^1,x',t)}{h} \\ &= -D^h_t\hat{w}(x^1,x',t), \end{aligned}$$

for any $(x,t) \in Q_{3R}$, we have

$$D^h_t\hat{w} \text{ is the odd extension of } D^h_t w \text{ from } Q^+_{3R} \text{ to } Q_{3R}. \tag{3.74}$$

Also let $\hat{a}^{t,h}_{ij}(x,t)$ be an extension of $a^{t,h}_{ij}(x,t)$ from Q^+_{3R} to Q_{3R} such that

$$\begin{cases} \hat{a}^{t,h}_{11}(-x^1,x',t) = a^{t,h}_{11}(x^1,x',t), & \\ \hat{a}^{t,h}_{ij}(-x^1,x',t) = a^{t,h}_{ij}(x^1,x',t) & \text{for } 1<i\le n, 1<j\le n, \\ \hat{a}^{t,h}_{1j}(-x^1,x',t) = -a^{t,h}_{1j}(x^1,x',t) & \text{for } 1<j\le n, \\ \hat{a}^{t,h}_{i1}(-x^1,x',t) = -a^{t,h}_{i1}(x^1,x',t) & \text{for } 1<i\le n, \end{cases} \tag{3.75}$$

for $(x^1,x') \in Q^+_{3R}$. One can directly check that $a^{t,h}_{ij}(x,t)$ satisfy

$$\begin{cases} |\hat{a}^{t,h}_{ij}(x,t)| \le \Lambda, \\ \hat{a}^{t,h}_{ij}(x,t)\zeta_i\zeta_j \ge \lambda|\zeta|^2, \end{cases} \tag{3.76}$$

for a.e. $(x,t) \in Q_{3R}$ and every $\zeta \in \mathbb{R}^n$. Since $D^h_t\hat{w} = 0$ on T_{3R}, one can directly check from (3.73), (3.74) and (3.75) that

$$0 = (D^h_t\hat{w})_t - D_i[\hat{a}^{t,h}_{ij}(x,t)D_j(D^h_t\hat{w})] \quad \text{in } Q_{3R}. \tag{3.77}$$

Lemma 3.3.11. *If w is a weak solution of (3.62), then we have*

$$\int_{Q_\rho(x_0,t_0)} |DD^h_t\hat{w}|^2\,dxdt \le \frac{C}{(r-\rho)^2}\int_{Q_r(x_0,t_0)} |D^h_t\hat{w}|^2\,dxdt.$$

for any $(x_0,t_0) \in Q_R$, $0<\rho<r\le 2R$ and $0<h<R^2$.

Proof. Choose a cut-off function $\eta \in C^\infty_c(Q_r(x_0,t_0))$ such that

$$0\le\eta\le 1, \eta = 1 \text{ in } Q_\rho(x_0,t_0), |D\eta|^2 \le \frac{C}{r-\rho} \text{ and } |\eta_t| \le \frac{C}{(r-\rho)^2}. \tag{3.78}$$

Let $0<l<R^2$. Use Steklov average formulation on (3.77), and then test by $\varphi = \eta^2[D^h_t\hat{w}]_l$ to find that

$$0 = \int_{Q_r(x_0,t_0)} ([D^h_t\hat{w}]_l)_t\eta^2[D^h_t\hat{w}]_l + [\hat{a}^{t,h}_{ij}(x,t)D_j(D^h_t\hat{w})]_l D_i(\eta^2[D^h_t\hat{w}]_l)\,dxdt. \tag{3.79}$$

Also from the fact that $\eta \in C_c^\infty(Q_r(x_0,t_0))$, a direct calculation implies

$$\begin{aligned} &\int_{Q_r(x_0,t_0)} ([D_t^h \hat{w}]_l)_t \eta^2 [D_t^h \hat{w}]_l \, dxdt \\ &\quad = \int_{Q_r(x_0,t_0)} \frac{(|[D_t^h \hat{w}]_l|^2 \eta^2)_t}{2} - |[D_t^h \hat{w}]_l|^2 \eta \eta_t \, dxdt \\ &\quad = -\int_{Q_r(x_0,t_0)} |[D_t^h \hat{w}]_l|^2 \eta \eta_t \, dxdt. \end{aligned} \tag{3.80}$$

By sending $l \to 0$, we have from (3.79) and (3.80) that

$$0 = \int_{Q_r(x_0,t_0)} |D_t^h \hat{w}|^2 \eta \eta_t - \hat{a}_{ij}^{t,h}(x,t) D_j(D_t^h \hat{w}) D_i(\eta^2 D_t^h \hat{w}) \, dxdt. \tag{3.81}$$

From (3.81), a direct calculation gives

$$\begin{aligned} &\int_{Q_r(x_0,t_0)} \hat{a}_{ij}^{t,h}(x,t) D_j(D_t^h \hat{w}) \eta^2 D_i(D_t^h \hat{w}) \, dxdt \\ &\quad = \int_{Q_r(x_0,t_0)} \hat{a}_{ij}^{t,h}(x,t) D_j(D_t^h \hat{w}) [D_i(\eta^2 D_t^h \hat{w}) - 2\eta D_i \eta (D_t^h \hat{w})] \, dxdt \\ &\quad = \int_{Q_r(x_0,t_0)} |D_t^h \hat{w}|^2 \eta \eta_t - \hat{a}_{ij}^{t,h}(x,t) D_j(D_t^h \hat{w}) 2\eta D_i \eta (D_t^h \hat{w}) \, dxdt. \end{aligned} \tag{3.82}$$

We apply (3.71) and Young's inequality to (3.82). Then the choice of the cut-off function (3.78) implies that

$$\begin{aligned} &\int_{Q_r(x_0,t_0)} |DD_t^h \hat{w}|^2 \eta^2 \, dxdt \\ &\quad \le C \int_{Q_r(x_0,t_0)} |D_t^h \hat{w}|^2 (|D\eta|^2 + |\eta D_t \eta|) \, dxdt \\ &\quad \le \frac{C}{(r-\rho)^2} \int_{Q_r(x_0,t_0)} |D_t^h \hat{w}|^2 \, dxdt. \end{aligned} \tag{3.83}$$

Thus (3.78) and (3.83) imply that the lemma holds. □

Let $\hat{a}$ be the even extension of $a^1(Dw(x,t), x^1)$ from Q_{4R}^+ to Q_{4R}. By using

(3.65) and the constant ν_0 in Lemma 3.2.3, we write

$$J(x,t) = (\nu_0 \lambda^{-1} \hat{a}(x,t), D_2 \hat{w}(x,t), \cdots, D_n \hat{w}(x,t)) \quad ((x,t) \in Q_{4R}). \tag{3.84}$$

Then from Lemma 3.2.3 and (3.65), we have

$$\begin{cases} |Dw(x,t)| \le C|J(x,t)| \le C|Dw(x,t)| & ((x,t) \in Q_{4R}^+), \\ |J(-x^1, x', t)| = |J(x^1, x', t)| & ((x^1, x', t) \in Q_{4R}). \end{cases} \tag{3.85}$$

Lemma 3.3.12. *If w is a weak solution of (3.62), then we have $w_t \in L^2_{loc}(Q_{2R}(x_0,t_0))$ with the estimate*

$$\int_{Q_\rho(x_0,t_0)} |\hat{w}_t|^2 \, dxdt \le \frac{C}{(r-\rho)^2} \int_{Q_r(x_0,t_0)} |J|^2 \, dxdt,$$

for any $(x_0,t_0) \in Q_R$ and $0 < \rho < r \le 2R$.

Proof. Let $d_0 = \dfrac{2\rho + r}{3}$, $d_\infty = \dfrac{\rho + 2r}{3}$ and $0 < h < r^2 - d_\infty^2$. For $m \ge 1$, set

$$d_m = d_0 + \sum_{l=1}^{m} \frac{r-\rho}{3 \cdot 2^l} \quad \text{and} \quad e_m = \frac{d_m + d_{m+1}}{2}.$$

Choose a cut-off function $\eta \in C_c^\infty(Q_{e_m}(x_0,t_0))$ such that

$$0 \le \eta \le 1, \eta = 1 \text{ in } Q_{d_m(x_0,t_0)}, |D\eta| \le \frac{C2^m}{r-\rho} \text{ and } |\eta_t| \le \frac{C4^m}{(r-\rho)^2}. \tag{3.86}$$

Since $w = 0$ on T_{3R}, we have $D_t^h w = 0$ on T_{3R}. Use Steklov average formulation on (3.62) and then test by $\eta^2 D_t^h w$ to find that

$$\begin{aligned} &\int_{Q_{3R}^+ \cap Q_{e_m}(x_0,t_0)} ([w]_h)_t \eta^2 (D_t^h w) \, dxdt \\ &\quad = -\int_{Q_{3R}^+ \cap Q_{e_m}(x_0,t_0)} [a^i(Dw, x^1)]_h D_i[\eta^2 D_t^h w] \, dxdt. \end{aligned} \tag{3.87}$$

From (3.12), we have $([w]_h)_t = D_t^h w$. Thus (3.87) and Young's inequality imply that

$$
\begin{aligned}
&\int_{Q_{3R}^{+}\cap Q_{e_m}(x_0,t_0)} \eta^2 |D_t^h w|^2 \, dxdt \\
&\quad \le \int_{Q_{3R}^{+}\cap Q_{e_m}(x_0,t_0)} \kappa |D_t^h Dw|^2 \eta^2 + \frac{\eta^2 |D_t^h w|^2}{4} \, dxdt \\
&\qquad + C \int_{Q_{3R}^{+}\cap Q_{e_m}(x_0,t_0)} \left(\frac{\eta^2}{\kappa} + |D\eta|^2 \right) |[a^i(Dw, x^1)]_h|^2 \, dxdt.
\end{aligned} \tag{3.88}
$$

Now, take $\rho = e_m$ and $r = d_{m+1}$ in Lemma 3.3.11 to find that

$$
\begin{aligned}
&\int_{Q_{3R}^{+}\cap Q_{e_m}(x_0,t_0)} |D_t^h Dw|^2 \, dxdt \\
&\quad \le \frac{C_2 4^m}{(r-\rho)^2} \int_{Q_{3R}^{+}\cap Q_{d_{m+1}}(x_0,t_0)} |D_t^h w|^2 \, dxdt.
\end{aligned} \tag{3.89}
$$

Take κ so that $\dfrac{C_2 \kappa 4^m}{(r-\rho)^2} = \dfrac{1}{18}$. By combining (3.86), (3.88) and (3.89), we have

$$
\begin{aligned}
&\int_{Q_{3R}^{+}\cap Q_{d_m}(x_0,t_0)} |D_t^h w|^2 \, dxdt \\
&\quad \le \int_{Q_{3R}^{+}\cap Q_{d_{m+1}}(x_0,t_0)} \frac{|D_t^h w|^2}{9} + \frac{C 4^m |[a^i(Dw, x^1)]_h|^2}{(r-\rho)^2} \, dxdt.
\end{aligned} \tag{3.90}
$$

Multiply (3.90) by $\dfrac{1}{9^m}$ and sum it from $m = 0$ to ∞. Then we see that

$$
\begin{aligned}
&\sum_{m=0}^{\infty} \frac{1}{9^m} \int_{Q_{3R}^{+}\cap Q_{d_m}(x_0,t_0)} |D_t^h w|^2 \, dxdt \\
&\quad \le \sum_{m=0}^{\infty} \frac{1}{9^{m+1}} \int_{Q_{3R}^{+}\cap Q_{d_{m+1}}(x_0,t_0)} |D_t^h w|^2 \, dxdt \\
&\qquad + \sum_{m=0}^{\infty} \left(\frac{4}{9} \right)^m \frac{C}{(r-\rho)^2} \int_{Q_{3R}^{+}\cap Q_{d_{m+1}}(x_0,t_0)} |[a^i(Dw, x^1)]_h|^2 \, dxdt.
\end{aligned} \tag{3.91}
$$

Since $d_m \le d_\infty = \dfrac{\rho+2r}{3}$ for any $m \ge 1$ and $0 < h < r^2 - d_\infty^2$, (3.13) implies

$$\begin{aligned}
&\int_{Q_{3R}^+\cap Q_{d_m}(x_0,t_0)} |[a^i(Dw,x^1)]_h|^2 \, dxdt \\
&\quad \le \int_{Q_{3R}^+\cap Q_r(x_0,t_0)} |a^i(Dw,x^1)|^2 \, dxdt,
\end{aligned} \tag{3.92}$$

for any $m \ge 1$. Combine (3.91) and (3.92), and then use (3.17) and (3.85) find that

$$\begin{aligned}
&\int_{Q_{3R}^+\cap Q_{d_0}(x_0,t_0)} |D_t^h w|^2 dxdt \\
&\quad \le \frac{C}{(r-\rho)^2}\int_{Q_{3R}^+\cap Q_r(x_0,t_0)} |a^i(Dw,x^1)|^2 dxdt \\
&\quad \le \frac{C}{(r-\rho)^2}\int_{Q_{3R}^+\cap Q_r(x_0,t_0)} |Dw|^2 \, dxdt \\
&\quad \le \frac{C}{(r-\rho)^2}\int_{Q_{3R}^+\cap Q_r(x_0,t_0)} |J|^2 \, dxdt.
\end{aligned} \tag{3.93}$$

Recall from (3.74) that $D_t^h\hat{w}$ is the odd extension of $D_t^h w$. Since (3.93) holds for any $(x_0,t_0) \in Q_R$, (3.85) and (3.93) imply that

$$\begin{aligned}
\int_{Q_{3R}^-\cap Q_{d_0}(x_0,t_0)} |D_t^h\hat{w}|^2 \, dxdt &\le \int_{Q_{3R}^+\cap Q_{d_0}(-x_0^1,x_0',t_0)} |D_t^h\hat{w}|^2 \, dxdt \\
&\le \frac{C}{(r-\rho)^2}\int_{Q_{3R}^+\cap Q_r(-x_0^1,x_0',t_0)} |J|^2 \, dxdt \\
&\le \frac{C}{(r-\rho)^2}\int_{Q_{3R}^-\cap Q_r(x_0,t_0)} |J|^2 \, dxdt.
\end{aligned} \tag{3.94}$$

Since $Q_r(x_0,t_0) \subset Q_{3R}$ and $d_0 = \dfrac{2\rho+r}{3}$, we have from (3.93) and (3.94) that

$$\int_{Q_{\frac{2\rho+r}{3}}(x_0,t_0)} |D_t^h\hat{w}|^2 \, dxdt \le \frac{C}{(r-\rho)^2}\int_{Q_r(x_0,t_0)} |J|^2 \, dxdt. \tag{3.95}$$

Since $0 < h < r^2 - d_\infty^2$ was arbitrary chosen, we from (3.95) and Lemma

3.2.6 that $\hat{w}_t \in L^2(Q_\rho)$ with the following estimate

$$\int_{Q_\rho(x_0,t_0)} |\hat{w}_t|^2 \, dxdt \le \frac{C}{(r-\rho)^2} \int_{Q_r(x_0,t_0)} |J|^2 \, dxdt. \tag{3.96}$$

$\square$

Lemma 3.3.13. *If w is a weak solution of (3.62), then we have $D\hat{w}_t \in L^2_{loc}(Q_{2R}(x_0,t_0))$ with the following estimate*

$$\int_{Q_\rho(x_0,t_0)} |Dw_t|^2 \, dxdt \le \frac{C}{(r-\rho)^2} \int_{Q_r(x_0,t_0)} |\hat{w}_t|^2 \, dxdt,$$

for any $0 < \rho < r < 2R$.

Proof. From Lemma 3.3.12, we have $w_t \in L^2_{loc}(Q_{2R}(x_0,t_0))$. Then Lemma 3.3.11 implies that

$$\begin{aligned}\int_{Q_{\frac{2\rho+r}{3}}(x_0,t_0)} |DD_t^h \hat{w}|^2 \, dxdt &\le \frac{C}{(r-\rho)^2} \int_{Q_{\frac{\rho+2r}{3}}(x_0,t_0)} |D_t^h \hat{w}|^2 \, dxdt \\ &\le \frac{C}{(r-\rho)^2} \int_{Q_r(x_0,t_0)} |\hat{w}_t|^2 \, dxdt,\end{aligned}$$

for any $0 < h < r^2 - \left(\frac{2\rho+r}{3}\right)^2$ and $0 < \rho < r < 2R$. Then Lemma 3.2.6 implies that the lemma holds. $\square$

Lemma 3.3.14. *If w is a weak solution of (3.62), then we have*

$$⨍_{Q_\rho(x_0,t_0)} |\hat{w}_t - (\hat{w}_t)_{Q_\rho(x_0,t_0)}|^2 dxdt \le C \left(\frac{\rho}{r}\right)^{2\alpha} ⨍_{Q_r(x_0,t_0)} |\hat{w}_t|^2 dxdt,$$

for any $0 < \rho \le r < 2R$ and for some constant $\alpha = \alpha(n,\lambda,\Lambda) \in (0,1)$.

Proof. If $r \le 4\rho$, then we have from Lemma 3.2.1 that

$$\begin{aligned}⨍_{Q_\rho(x_0,t_0)} |\hat{w}_t - (\hat{w}_t)_{Q_\rho(x_0,t_0)}|^2 \, dxdt &\le C ⨍_{Q_\rho(x_0,t_0)} |\hat{w}_t|^2 \, dxdt \\ &\le C \left(\frac{\rho}{r}\right)^{2\alpha} ⨍_{Q_r(x_0,t_0)} |\hat{w}_t|^2 \, dxdt.\end{aligned}$$

Thus we see that the lemma holds when $r \le 4\rho$.

So suppose that $4\rho < r$. Let $0 < h < \frac{r^2}{4}$. From (3.77), we see that $D_t^h \hat{w}$ is a weak solution of

$$(D_t^h \hat{w})_t - D_i[\hat{a}_{ij}^{t,h}(x,t) D_j(D_t^h \hat{w})] = 0 \quad \text{in } Q_{3R}. \tag{3.97}$$

By using (3.71), apply Lemma 3.2.5 to (3.97). Then we have

$$\begin{aligned} &\fint_{Q_\rho(x_0,t_0)} |D_t^h \hat{w} - (D_t^h \hat{w})_{Q_\rho(x_0,t_0)}|^2 \, dxdt \\ &\quad \le C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_{\frac{r}{2}}(x_0,t_0)} |D_t^h \hat{w}|^2 \, dxdt. \end{aligned} \tag{3.98}$$

In view of Lemma 3.3.12, we have $w_t \in L^2_{loc}(Q_{2R}(x_0,t_0))$. Thus Lemma 3.2.6 implies that

$$\fint_{Q_{r/2}(x_0,t_0)} |D_t^h \hat{w}|^2 \, dxdt \le C \fint_{Q_r(x_0,t_0)} |\hat{w}_t|^2 \, dxdt \tag{3.99}$$

From Lemma 3.3.3, we have $w_t \in L^2_{loc}(Q_{2R}(x_0,t_0))$. Thus from Lemma 3.2.6, we have

$$D_t^h w \to D_t w \text{ in } L^2_{loc}(Q_{2R}(x_0,t_0)) \text{ as } h \to 0,$$

which implies

$$\begin{aligned} &\fint_{Q_\rho(x_0,t_0)} |\hat{w}_t - (\hat{w}_t)_{Q_{\rho(x_0,t_0)}}|^2 \, dxdt \\ &\quad \le \liminf_{h\to 0} \fint_{Q_\rho(x_0,t_0)} |D_t^h \hat{w} - (D_t^h \hat{w})_{Q_\rho(x_0,t_0)}|^2 \, dxdt. \end{aligned} \tag{3.100}$$

Thus we see from (3.98), (3.99) and (3.100) that the lemma holds. □

Since the estimate in Lemma 3.3.14 is invariant under translation and scaling, Lemma 3.2.2 implies the following lemma.

Lemma 3.3.15. *If w is a weak solution of (3.62), then we have*

$$\|\hat{w}_t\|_{L^\infty(Q_{\frac{r}{4}}(x_0,t_0))} \le C \left(\fint_{Q_{\frac{r}{2}}(x_0,t_0)} |\hat{w}_t|^2 \, dxdt \right)^{\frac{1}{2}},$$

for any $(x_0, t_0) \in Q_R$ *and* $0 < r \leq R$.

Proof. Choose any point $(x_0, t_0) \in Q_{\frac{r}{4}}$. Since Lemma 3.3.14 is invariant under translation and scaling, we have

$$\begin{aligned} \frac{1}{\rho^{2\alpha}} \fint_{Q_\rho(x_0,t_0)} \left|\hat{w}_t - (\hat{w}_t)_{Q_\rho(x_0,t_0)}\right|^2 \, dxdt &\leq \frac{C}{r^{2\alpha}} \fint_{Q_{\frac{r}{4}}(x_0,t_0)} |\hat{w}_t|^2 \, dxdt \\ &\leq \frac{C}{r^{2\alpha}} \fint_{Q_{\frac{r}{2}}(x_0,t_0)} |\hat{w}_t|^2 \, dxdt, \end{aligned} \tag{3.101}$$

for any $0 < \rho \leq \frac{r}{4}$ and for some constant $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$. Thus from (3.101) and Lemma 3.2.2, we have

$$\|\hat{w}_t\|_{L^\infty(Q_{\frac{r}{4}}(x_0,t_0))} \leq C \left(\fint_{Q_{\frac{r}{2}}(x_0,t_0)} |\hat{w}_t|^2 \, dxdt \right)^{\frac{1}{2}}.$$

$\square$

Lemma 3.3.16. *If w is a weak solution of (3.62), then we have*

$$\fint_{Q_\rho(x_0,t_0)} \left|\hat{a} - (\hat{a})_{Q_\rho(x_0,t_0)}\right|^2 \, dxdt \leq C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r(x_0,t_0)} |J|^2 \, dxdt,$$

for any $(x_0, t_0) \in Q_R$, $0 < \rho \leq r \leq R$ *and for some constant* $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$.

Proof. If $r \leq 8\rho$, then we have $r \leq 8\rho \leq 8r$, and from the next inequality

$$\begin{aligned} \fint_{Q_\rho(x_0,t_0)} \left|\hat{a} - (\hat{a})_{Q_\rho(x_0,t_0)}\right|^2 \, dxdt &\leq C \fint_{Q_\rho(x_0,t_0)} |\hat{a}|^2 \, dxdt \\ &\leq C \fint_{Q_\rho(x_0,t_0)} |J|^2 \, dxdt \\ &\leq C \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r(x_0,t_0)} |J|^2 \, dxdt, \end{aligned}$$

we see that Lemma 3.3.16 holds. So we assume that $0 < 8\rho < r \leq R$.

Let $1 < k \leq n$. Recall that $a^1(\xi, x^1)$ does not depend on x^k-variable and t-variable. Since $\hat{a}(x, t)$ is the even extension of $a^1(Dw(x, t), x^1)$ from Q^+_{3R} to

Q_{3R}, we have

$$\begin{cases} D_k\hat{a}(x,t) = a_{1j}(x,t)D_{jk}w(x,t), \\ \partial_t\hat{a}(x,t) = a_{1j}(x,t)D_j w_t(x,t), \end{cases} \tag{3.102}$$

for any $(x,t) \in Q_{3R}^+$. By the definition of weak derivatives, (3.62) implies that

$$\begin{aligned} D_1[\hat{a}(x,t)] &= D_1[a^1(Dw(x,t),x^1)] \\ &= w_t(x,t) - \sum_{1<i\le n} D_i[a^i(Dw(x,t),x^1)] \\ &= w_t(x,t) - \sum_{1<i\le n} a_{ij}(x,t)D_{ij}w(x,t), \end{aligned} \tag{3.103}$$

for any $(x,t) \in Q_{3R}^+$. Since $\hat{a}(x,t)$ is an even extension of $a^1(Dw(x,t),x^1)$, we have $D\hat{a}$ and $\hat{a}_t$ exist in Q_{3R} and

$$\begin{cases} D_k\hat{a}(x,t) = D_k\hat{a}(-x^1,x',t), \\ \partial_t\hat{a}(x,t) = \partial_t\hat{a}(-x^1,x',t), \\ D_1\hat{a}(x,t) = -D_1\hat{a}(-x^1,x',t), \end{cases} \tag{3.104}$$

for any $(x^1,x',t) \in Q_{3R}^-$. Thus from (3.102), (3.103) and (3.104), we have

$$\begin{cases} D_k\hat{a}(x,t) = a_{1j}(-x^1,x',t)D_{jk}w(-x^1,x',t), \\ \partial_t\hat{a}(x,t) = a_{1j}(-x^1,x',t)D_j w_t(-x^1,x',t), \\ D_1\hat{a}(x,t) = -w_t(-x^1,x',t) + \displaystyle\sum_{1<i\le n} a_{ij}(-x^1,x',t)D_{ij}w(-x^1,x',t), \end{cases} \tag{3.105}$$

for any $(x,t) \in Q_{3R}^-$. Recall from (3.64) and (3.65) that $\hat{w}$ and $D_k\hat{w}$ are the odd extensions of w and $D_k w$ from Q_{3R}^+ to Q_{3R} respectively. Thus we have

$$\begin{cases} |w_t(-x^1,x',t)| = |\hat{w}_t(-x^1,x',t)| = |\hat{w}_t(x^1,x',t)|, \\ |DD_k w(-x^1,x',t)| = |DD_k\hat{w}(-x^1,x',t)| = |DD_k\hat{w}(x^1,x',t)|, \end{cases} \tag{3.106}$$

for any $(x,t) \in Q_{3R}^-$ and $1 < k \le n$. Thus from (3.105) and (3.106), we have

$$\begin{aligned}
\fint_{Q_\rho(x_0,t_0)\cap Q_{3R}^-} |D\hat{a}|^2 \, dxdt &\le C \fint_{Q_\rho(x_0,t_0)\cap Q_{3R}^-} |w_t(-x^1,x',t)|^2 \\
&\qquad + \sum_{1<k\le n} |DD_k w(-x^1,x',t)|^2 \, dxdt \\
&\le C \fint_{Q_\rho(x_0,t_0)} |\hat{w}_t|^2 + \sum_{1<k\le n} |DD_k\hat{w}|^2 dxdt.
\end{aligned} \tag{3.107}$$

Since $8\rho < r$, we have from Lemma 3.3.15 that

$$\fint_{Q_{2\rho}(x_0,t_0)} |\hat{w}_t|^2 \, dxdt \le C\|\hat{w}_t\|^2_{L^\infty(Q_{\frac{r}{4}}(x_0,t_0))} \le C \fint_{Q_{\frac{r}{2}}(x_0,t_0)} |\hat{w}_t|^2 \, dxdt. \tag{3.108}$$

In view of (3.102), (3.103) and (3.107), we have

$$\begin{aligned}
\fint_{Q_\rho(x_0,t_0)} |D\hat{a}|^2 \, dxdt &= \fint_{Q_\rho(x_0,t_0)\cap Q_{3R}^+} + \fint_{Q_\rho(x_0,t_0)\cap Q_{3R}^-} |D\hat{a}|^2 \, dxdt \\
&\le C \fint_{Q_\rho(x_0,t_0)} |\hat{w}_t|^2 + \sum_{1<k\le n} |DD_k\hat{w}|^2 \, dxdt.
\end{aligned} \tag{3.109}$$

From Lemma 3.3.10 and (3.85), we have

$$\begin{aligned}
\fint_{Q_\rho(x_0,t_0)} |DD_k\hat{w}|^2 dxdt &\le \frac{C}{\rho^2} \fint_{Q_{2\rho}(x_0,t_0)} |D_k\hat{w} - (D_k\hat{w})_{Q_{2\rho}}|^2 \, dxdt \\
&\le \frac{C}{\rho^2}\left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r(x_0,t_0)} |D_k\hat{w}|^2 \, dxdt \\
&\le \frac{C}{\rho^2}\left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r(x_0,t_0)} |J|^2 \, dxdt.
\end{aligned} \tag{3.110}$$

By taking $\rho = \frac{r}{2}$ in Lemma 3.3.12, we have from (3.108) that

$$\fint_{Q_{2\rho}(x_0,t_0)} |\hat{w}_t|^2 \, dxdt \le C \fint_{Q_{\frac{r}{2}}(x_0,t_0)} |\hat{w}_t|^2 \, dxdt \le \frac{C}{r^2} \fint_{Q_r(x_0,t_0)} |J|^2 \, dxdt. \tag{3.111}$$

Since $\alpha \in (0,1)$, we have from (3.109), (3.110) and (3.111) that

$$\fint_{Q_\rho(x_0,t_0)} \rho^2 \left|D\hat{a}\right|^2 \ dxdt \le C\left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r(x_0,t_0)} |J|^2 \ dxdt. \tag{3.112}$$

From (3.102), (3.105) and Lemma 3.3.13, we have

$$\fint_{Q_\rho(x_0,t_0)} |\hat{a}_t|^2 dxdt \le C \fint_{Q_\rho(x_0,t_0)} |D\hat{w}_t|^2 dxdt \le \frac{C}{\rho^2} \fint_{Q_{2\rho}(x_0,t_0)} |\hat{w}_t|^2 \ dxdt. \tag{3.113}$$

Since $\alpha \in (0,1)$, we have from (3.111), (3.112) and (3.113) that

$$\fint_{Q_\rho(x_0,t_0)} \rho^2 \left|D\hat{a}\right|^2 + \rho^4 \left|\hat{a}_t\right|^2 dxdt \le C\left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r(x_0,t_0)} |J|^2 dxdt. \tag{3.114}$$

Then we use the Poincaré inequality and a scaling argument to find that

$$\begin{aligned} &\fint_{Q_\rho(x_0,t_0)} \left|\hat{a} - (\hat{a})_{Q_\rho(x_0,t_0)}\right|^2 \ dxdt \\ &\quad \le C \fint_{Q_\rho(x_0,t_0)} \rho^2 \left|D\hat{a}\right|^2 + \rho^4 \left|\hat{a}_t\right|^2 \ dxdt. \end{aligned} \tag{3.115}$$

From (3.114) and (3.115), we see that the lemma holds. $\square$

Recall the definition of $J(x,t)$ in (3.84). Then we have the following lemma.

Lemma 3.3.17. *Let $(x_0,t_0) \in Q_R$. If w is a weak solution of (3.62), then we have*

$$\fint_{Q_\rho(x_0,t_0)} \left|J - (J)_{Q_\rho(x_0,t_0)}\right|^2 dxdt \le C\left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r(x_0,t_0)} |J|^2 dxdt,$$

for any $0 < \rho \le r \le 3R$ and for some constant $\alpha = \alpha(n,\lambda,\Lambda) \in (0,1)$.

Proof. Recall the definition of $J(x,t)$ in (3.84). We prove the lemma by considering four cases: $r \le 8\rho$, $8\rho < r \le R$, $8\rho < R < r$ and $R \le 8\rho < r$.

From Lemma 3.2.1, we have

$$\begin{aligned}\fint_{Q_\rho(x_0,t_0)} \left|J-(J)_{Q_\rho(x_0,t_0)}\right|^2\ dxdt &\le \fint_{Q_\rho(x_0,t_0)} |J|^2\ dxdt \\ &\le C\fint_{Q_r(x_0,t_0)} |J|^2\ dxdt,\end{aligned} \tag{3.116}$$

for any $r \le 8\rho$. From Lemma 3.3.10 and Lemma 3.3.16, we have

$$\fint_{Q_\rho(x_0,t_0)} \left|J-(J)_{Q_\rho}\right|^2\ dxdt \le C\left(\frac{\rho}{r}\right)^{2\alpha}\fint_{Q_r(x_0,t_0)} |J|^2\, dxdt, \tag{3.117}$$

for any $8\rho < r \le R$. If $R < r$, then we have $R < r \le 3R$, and Lemma 3.3.10 and Lemma 3.3.16 imply that

$$\begin{aligned}\fint_{Q_\rho(x_0,t_0)} \left|J-(J)_{Q_\rho}\right|^2 dxdt &\le C\left(\frac{\rho}{R}\right)^{2\alpha}\fint_{Q_R(x_0,t_0)} |J|^2 dxdt \\ &\le C\left(\frac{\rho}{r}\right)^{2\alpha}\fint_{Q_r(x_0,t_0)} |J|^2 dxdt,\end{aligned} \tag{3.118}$$

for any $8\rho < R < r$. If $R \le 8\rho < r$, then we have $R \le 8\rho < r \le 3R$ which implies $r \le 24\rho$. Thus Lemma 3.2.1 implies that

$$\begin{aligned}\fint_{Q_\rho(x_0,t_0)} \left|J-(J)_{Q_\rho(x_0,t_0)}\right|^2\ dxdt &\le \fint_{Q_\rho(x_0,t_0)} |J|^2\ dxdt \\ &\le C\fint_{Q_r(x_0,t_0)} |J|^2\ dxdt,\end{aligned} \tag{3.119}$$

for any $R \le 8\rho < r$. From (3.116), (3.117), (3.118) and (3.119), we see that the lemma holds. $\square$

Now, we prove the follow Lipschitz regularity of w.

Lemma 3.3.18. *If w is a weak solution of (3.62), w is Lipschitz continuous in Q_R^+ with the estimate*

$$\|Dw\|_{L^\infty(Q_R^+)} \le C\|Dw\|_{L^2(Q_{4R}^+)}.$$

Proof. Since Lemma 3.3.17 is invariant under translation and scaling, Lemma

3.3.17 implies

$$
\begin{aligned}
\frac{1}{\rho^{2\alpha}} \fint_{Q_\rho(x_0,t_0)} \left| J - (J)_{Q_\rho(x_0,t_0)} \right|^2 dxdt &\leq \frac{C}{R^{2\alpha}} \fint_{Q_R(x_0,t_0)} |J|^2 \, dxdt \\
&\leq \frac{C}{R^{2\alpha}} \fint_{Q_{4R}} |J|^2 \, dxdt,
\end{aligned} \tag{3.120}
$$

for any $(x_0, t_0) \in Q_R$, $0 < \rho \leq R$ and for some constant $\alpha = \alpha(n, \lambda, \Lambda) \in (0,1)$. Thus from (3.120) and Lemma 3.2.2, we have

$$
\begin{aligned}
\|J\|_{L^\infty(Q_R)} &\leq C \left[\fint_{Q_R} |J|^2 dxdt \right]^{\frac{1}{2}} + CR^\alpha [J]_{L^{2,\alpha}(Q_R)} \\
&\leq C \left[\fint_{Q_{4R}} |J|^2 dxdt \right]^{\frac{1}{2}}.
\end{aligned} \tag{3.121}
$$

In view of (3.85), we have

$$
\|Dw\|_{L^\infty(Q_R^+)} \leq \|J\|_{L^\infty(Q_R^+)} \leq \|J\|_{L^\infty(Q_R)}, \tag{3.122}
$$

and

$$
\left[\fint_{Q_{4R}} |J|^2 dxdt \right]^{\frac{1}{2}} \leq C \left[\fint_{Q_{4R}^+} |J|^2 dxdt \right]^{\frac{1}{2}} \leq C \left[\fint_{Q_{4R}^+} |Dw|^2 dxdt \right]^{\frac{1}{2}}. \tag{3.123}
$$

Thus from (3.121), (3.122) and (3.123), we see that the lemma holds. $\square$

3.4 Comparison estimates

We derive comparison estimates by using the method in [11, 3]. Comparison estimate is a standard method, and the method in [11, 3] even works for our problem.

3.4.1 Interior comparison estimates

We derive interior comparison estimates. Under the assumptions (3.1) and (3.2), let u be a weak solution of

$$u_t - \text{div } a(Du, x, t) = \text{div } F \text{ in } Q_6, \tag{3.124}$$

with

$$\fint_{Q_6} |Du|^2 \, dxdt \leq 1, \tag{3.125}$$

$$\fint_{Q_6} |F|^2 \, dxdt \leq \delta^2, \tag{3.126}$$

and

$$\fint_{Q_6} |\theta(a, \bar{a})|^2 \, dxdt \leq \delta^2, \tag{3.127}$$

for some $\bar{a}(\xi, x^1)$ satisfies (3.1), (3.2), where $\delta > 0$ to be determined.

Let v be the weak solution to its homogeneous problem

$$\begin{cases} v_t - \text{div } a(Dv, x, t) &= 0 \quad \text{in } Q_5, \\ v &= u \quad \text{on } \partial_p Q_5. \end{cases} \tag{3.128}$$

For any $\epsilon > 0$, we see from a comparison estimate that there exists a constant $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ such that

$$\fint_{Q_5} |D(u - v)|^2 \, dxdt \leq C \fint_{Q_5} |F|^2 \, dxdt \leq C\delta^2 \leq \epsilon^2. \tag{3.129}$$

We have from (3.125) and (3.126) that

$$\fint_{Q_5} |Dv|^2 \, dxdt \leq C. \tag{3.130}$$

By a well known higher integrability result, see for instance [25, 34], we have from (3.130) that

$$\fint_{Q_4} |Dv|^{2+\sigma_1} \, dxdt \leq C, \tag{3.131}$$

for some constant $\sigma_1 = \sigma_1(n, \lambda, \Lambda)$. Let w be the weak solution to the equation

$$\begin{cases} w_t - \text{div } \bar{a}(Dw, x^1) &= 0 \quad \text{in } Q_4, \\ w &= v \quad \text{on } \partial_p Q_4. \end{cases} \tag{3.132}$$

Lemma 3.4.1. *For any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ such that if v and w are the weak solution to (3.128) and (3.132) respectively, then we have*

$$\fint_{Q_4} |D(v-w)|^2 \, dxdt \leq \epsilon^2. \tag{3.133}$$

Proof. Test (3.128) and (3.132) by $v - w$ to find that

$$\begin{aligned} I_1 &= \fint_{Q_4} \langle \bar{a}(Dv, x^1) - \bar{a}(Dw, x^1), Dv - Dw \rangle \, dxdt \\ &\leq \fint_{Q_4} \langle \bar{a}(Dv, x^1) - a(Dv, x, t), Dv - Dw \rangle \, dxdt = I_2. \end{aligned} \tag{3.134}$$

From the ellipticity condition (3.2), we have

$$\lambda \fint_{Q_4} |Dv - Dw|^2 \, dxdt \leq I_1. \tag{3.135}$$

From Young's inequality, we have

$$\begin{aligned} I_2 &\leq \fint_{Q_4} \langle \bar{a}(Dv, x^1) - a(Dv, x, t), Dv - Dw \rangle \, dxdt \\ &\leq \fint_{Q_4} |\theta(a, \bar{a})||Dv||Dv - Dw| \, dxdt \\ &\leq \frac{\lambda}{2} \fint_{Q_4} |Dv - Dw|^2 \, dxdt + C(\lambda) \fint_{Q_4} |\theta(a, \bar{a})|^2 |Dv|^2 \, dxdt. \end{aligned} \tag{3.136}$$

Thus by combining (3.135) and (3.136), we have

$$\fint_{Q_4} |Dv - Dw|^2 \, dxdt \leq C \fint_{Q_4} |\theta(a, \bar{a})|^2 |Dv|^2 \, dxdt. \tag{3.137}$$

Note that $|\theta(a, \bar{a})| \leq 2\Lambda$ is bounded. From (3.127) and the higher integrability

(3.131), we have

$$
\begin{aligned}
&\fint_{Q_4} |\theta(a,\bar{a})|^2 |Dv|^2 \, dxdt \\
&\quad \le \left(\fint_{Q_4} |\theta(a,\bar{a})|^{\frac{2(2+\sigma_1)}{\sigma_1}} \, dxdt \right)^{\frac{\sigma_1}{2+\sigma_1}} \left(\fint_{Q_4} |Dv|^{2+\sigma_1} \, dxdt \right)^{\frac{2}{2+\sigma_1}} \\
&\quad \le C\delta^{\sigma}.
\end{aligned}
\tag{3.138}
$$

By combining (3.137) and (3.138), we have

$$
\fint_{Q_4} |Dv - Dw|^2 \, dxdt \le C\delta^{\sigma}. \tag{3.139}
$$

Thus by taking $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ small enough, we see that the lemma holds. □

From Lemma 3.3.9, we have the following Lipschitz estimate

$$
\|Dw\|_{L^\infty(Q_3)} \le C \left(\fint_{Q_4} |Dw|^2 \, dxdt \right)^{\frac{1}{2}}. \tag{3.140}
$$

Thus the following conclusion holds from (3.129), (3.140) and Lemma 3.4.1.

Lemma 3.4.2. *There exists a constant $n_1 = n_1(n, \lambda, \Lambda)$ such that the following holds. For any $\epsilon > 0$, one can find a small $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ so that for such small δ, if u is a weak solution of (3.128), then there exists a weak solution w of (3.132) such that*

$$
\fint_{Q_3} |D(u-w)|^2 \, dxdt \le \epsilon^2 \quad \text{and} \quad \|Dw\|_{L^\infty(Q_3)} \le n_1. \tag{3.141}
$$

3.4.2 Boundary comparison estimates

We derive boundary comparison estimates. Assume that

$$
Q_6^+ \subset K_6 = Q_6 \cap \Omega_T \subset Q_6 \cap \{x : x^1 > -12\delta\}, \tag{3.142}
$$

Under the assumptions (3.1) and (3.2), let u be a weak solution of

$$\begin{cases} u_t - \text{div } a(Du, x, t) &= \text{div } F \quad \text{in } K_6, \\ u &= 0 \quad \text{on } \partial_w K_6, \end{cases} \tag{3.143}$$

with

$$\fint_{K_6} |Du|^2 \, dxdt \leq 1, \tag{3.144}$$

$$\fint_{K_6} |F|^2 \, dxdt \leq \delta^2, \tag{3.145}$$

and

$$\fint_{Q_6} |\theta(a, \bar{a})|^2 \, dxdt \leq \delta^2, \tag{3.146}$$

for some $\bar{a}(\xi, x^1)$ satisfies (3.1) and (3.2), where $\delta > 0$ to be determined.

Let v be the weak solution to its homogeneous problem

$$\begin{cases} v_t - \text{div } a(Dv, x, t) &= 0 \quad \text{in } K_6, \\ v &= u \quad \text{on } \partial_p K_6. \end{cases} \tag{3.147}$$

For any $\epsilon > 0$, we see from a comparison estimate that there exists a constant $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ such that

$$\fint_{K_6} |D(u - v)|^2 \, dxdt \leq C \fint_{K_6} |F|^2 \, dxdt \leq C\delta^2 \leq \epsilon^2. \tag{3.148}$$

We have from (3.145) and (3.144) that

$$\fint_{K_6} |Dv|^2 \, dxdt \leq C. \tag{3.149}$$

By a well known higher integrability result [34], we have from (3.149) that

$$\fint_{K_5} |Dv|^{2+\sigma_2} \, dxdt \leq C, \tag{3.150}$$

for some constant $\sigma_2 = \sigma_2(n, \lambda, \Lambda)$. Let w be the weak solution to the equation

$$\begin{cases} w_t - \text{div } \bar{a}(Dw, x^1) &= 0 \quad \text{in } K_5, \\ w &= v \quad \text{on } \partial_p K_5. \end{cases} \tag{3.151}$$

Lemma 3.4.3. *For any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ such that if v and w are the weak solution to (3.147) and (3.151) respectively, then we have*

$$\fint_{K_5} |D(v-w)|^2 \, dxdt \le \epsilon^2. \tag{3.152}$$

Proof. Test (3.147) and (3.151) by $v - w$ to find that

$$\begin{aligned} I_1 &= \fint_{K_5} \langle \bar{a}(Dv, x^1) - \bar{a}(Dw, x^1), Dv - Dw \rangle \, dxdt \\ &\le \fint_{K_5} \langle \bar{a}(Dv, x^1) - a(Dv, x, t), Dv - Dw \rangle \, dxdt = I_2. \end{aligned} \tag{3.153}$$

From the ellipticity condition (3.2), we have

$$\lambda \fint_{K_5} |Dv - Dw|^2 \, dxdt \le I_1. \tag{3.154}$$

From Young's inequality, we have

$$\begin{aligned} I_2 &\le \fint_{K_5} \langle \bar{a}(Dv, x^1) - a(Dv, x, t), Dv - Dw \rangle \, dxdt \\ &\le \fint_{K_5} |\theta(a, \bar{a})||Dv||Dv - Dw| \, dxdt \\ &\le \frac{\lambda}{2} \fint_{K_5} |Dv - Dw|^2 \, dxdt + C(\lambda) \fint_{K_5} |\theta(a, \bar{a})|^2 |Dv|^2 \, dxdt. \end{aligned} \tag{3.155}$$

Thus by combining (3.154) and (3.155), we have

$$\fint_{K_5} |Dv - Dw|^2 \, dxdt \le C \fint_{K_5} |\theta(a, \bar{a})|^2 |Dv|^2 \, dxdt. \tag{3.156}$$

From (3.142), (3.146) and the higher integrability (3.150), we have

$$\begin{aligned} &\fint_{K_5} |\theta(a, \bar{a})|^2 |Dv|^2 \, dxdt \\ &\le \left(\fint_{K_5} |\theta(a, \bar{a})|^{\frac{2(2+\sigma_2)}{\sigma_2}} \, dxdt \right)^{\frac{\sigma_2}{2+\sigma_2}} \left(\fint_{K_5} |Dv|^{2+\sigma_2} \, dxdt \right)^{\frac{2}{2+\sigma_2}} \\ &\le C\delta^{\sigma}. \end{aligned} \tag{3.157}$$

By combining (3.156) and (3.157), we have

$$\fint_{K_5} |Dv - Dw|^2 \, dxdt \leq C\delta^{\sigma}. \tag{3.158}$$

Thus by taking $\delta = \delta(\epsilon, n, \lambda, \Lambda)$ small enough, we see that the lemma holds. □

Let h be a weak solution to the reference problem

$$\begin{cases} h_t - \operatorname{div} \bar{a}(Dh, x^1) &= 0 \quad \text{in } Q_5^+, \\ h &= 0 \quad \text{on } Q_5 \cap \{x^1 = 0\}. \end{cases} \tag{3.159}$$

For any weak solution h of (3.159), the following Lipschitz estimate holds from Lemma 3.3.9 and Lemma 3.3.18.

$$\|Dh\|_{L^\infty(Q_4^+)} \leq C \left(\fint_{Q_5^+} |Dh|^2 \, dxdt \right)^{\frac{1}{2}}. \tag{3.160}$$

For comparing w and h, we use an approach in [3, Lemma 3.8].

Lemma 3.4.4. *For any $\epsilon > 0$, one can find a small $\delta = \delta(\epsilon, n, \lambda, \Lambda) > 0$ so that for such small $\delta > 0$, if w is a weak solution of (3.143) such that*

$$\fint_{K_5} |Dw|^2 \, dxdt \leq 1, \tag{3.161}$$

then there exists a weak solution h of (3.159) such that

$$\fint_{K_3} |D(w - \bar{h})|^2 \, dxdt \leq \epsilon^2 \quad \text{and} \quad \fint_{Q_5^+} |D\bar{h}|^2 \, dxdt \leq 1, \tag{3.162}$$

where $\bar{h}$ is the zero extension of h from Q_5^+ to Q_5.

Proof. We argue by contradiction. If not, there exist $\epsilon_0 > 0$, $\{w_m\}_{m=1}^\infty$ and $\{K_5^m\}_{m=1}^\infty$ such that w_m is a weak solution of

$$\begin{cases} (w_m)_t - \operatorname{div} \bar{a}(Dw_m, x^1) &= 0 \quad \text{in } K_5^m, \\ w &= 0 \quad \text{on } \partial_w K_5^m, \end{cases} \tag{3.163}$$

satisfying

$$\fint_{K_5^m} |Dw_m|^2 \, dxdt \leq 1, \tag{3.164}$$

and

$$Q_5^+ \subset K_5^m = \Omega_5^m \times (-4^2, 4^2) \subset Q_5 \cap \left\{ x^1 > -\frac{12}{m} \right\} \tag{3.165}$$

but

$$\fint_{K_3} |D(w_m - \bar{h})|^2 \, dxdt > \epsilon_0^2, \tag{3.166}$$

for any weak solution h of

$$\left\{ \begin{array}{rcll} h_t - \operatorname{div} \bar{a}(Dh, x^1) & = & 0 & \text{in } Q_5^+, \\ h & = & 0 & \text{on } Q_5 \cap \{x^1 = 0\}. \end{array} \right. \tag{3.167}$$

satisfying

$$\fint_{Q_5^+} |Dh|^2 \, dxdt \leq 1, \tag{3.168}$$

where $\bar{h}$ is the zero extension of h. From (3.164) and (3.165), we have

$$\fint_{Q_5^+} |Dw_m|^2 \, dxdt \leq C \fint_{K_5^m} |Dw_m|^2 \, dxdt \leq C, \tag{3.169}$$

and $\{w_m\}_{m=1}^{\infty}$ is uniformly bounded in $L^2(-5^2, 5^2; W^{1,p}(B_5^+))$. From (3.163), we see that $\{w_m\}_{m=1}^{\infty}$ is uniformly bounded in $L^2(-5^2, 5^2; W^{-1,2}(B_5^+))$. Consequently, one can apply Aubin-Lions Lemma to discover that there exists a subsequence of $\{w_m\}_{m=1}^{\infty}$, which we will still denote as $\{w_m\}_{m=1}^{\infty}$ and $w_0 \in W^2(-5^2, 5^2; B_5^+)$ such that

$$\left\{ \begin{array}{ll} Dw_m \rightharpoonup Dw_0 & \text{in } L^2(-5^2, 5^2; L^2(B_5^+)), \\ w_m \to w_0 & \text{in } L^2(-5^2, 5^2; L^2(B_5^+)), \\ (w_m)_t \rightharpoonup (w_0)_t & \text{in } L^2(-5^2, 5^2; W^{-1,2}(B_5^+)). \end{array} \right. \tag{3.170}$$

From (3.163), (3.164) and (3.165), one can prove that $w_0 = 0$ on $Q_5 \cap \{x^1 = 0\}$ in the trace sense. Then by letting $m \to \infty$ on (3.163), (3.164) and (3.165), one can use the method of Browder and Minty, see [11] for the details, to discover that w_0 is a weak solution of

$$\begin{cases} (w_0)_t - \text{div } \bar{a}(Dw_0, x^1) &= 0 \quad \text{in } Q_5^+, \\ w_0 &= 0 \quad \text{on } Q_5 \cap \{x^1 = 0\}. \end{cases} \tag{3.171}$$

On the other-hand, (3.164), (3.170) and the weak lower semicontinuity imply that

$$\fint_{Q_5^+} |Dw_0|^2 \, dxdt \leq \liminf_{m\to\infty} \fint_{Q_5^+} |Dw_m|^2 \, dxdt \leq 1. \tag{3.172}$$

Let $\bar{w}_m$ be the zero extension of w_m from K_5^m to Q_5. Then we have $\bar{w}_m = 0$ in $Q_5 \backslash K_5^m$, and we apply Sobolev-Poincaré's inequality [22, Theorem 3.16] in "slicewise" to find that

$$\|\bar{w}_m\|_{L^{2^*}(Q_4)} \leq C\|D\bar{w}_m\|_{L^2(Q_4)} = C\|Dw_m\|_{L^2(K_4^m)} \leq C,$$

which implies

$$\|w_m\|_{L^{2^*}(K_4^m)} \leq C, \tag{3.173}$$

for $2^* = \frac{2n}{n-2} > 2$. Let $\bar{w}_0$ be the zero extension of w_0 from Q_5^+ to Q_5. From (3.170), we have

$$\int_{Q_4^+} |w_m - \bar{w}_0|^2 \, dxdt \to 0 \text{ as } m \to \infty. \tag{3.174}$$

By using Hölder's inequality, we have from (3.173) that

$$\begin{aligned} &\int_{K_4^m \backslash Q_4^+} |w_m - \bar{w}_0|^2 \, dxdt \\ &\quad = \int_{K_4^m \backslash Q_4^+} |w_m|^2 \, dxdt \\ &\quad \leq \left(\int_{K_4^m \backslash Q_4^+} |w_m|^{2^*} \, dxdt \right)^{\frac{2}{2^*}} \left(\int_{K_4^m \backslash Q_4^+} 1 \, dxdt \right)^{\frac{2}{n+2}} \\ &\quad \leq \frac{C}{m^{\frac{2}{n+2}}} \quad \to 0 \text{ as } m \to \infty. \end{aligned} \tag{3.175}$$

By combining (3.174) and (3.175), we have

$$\int_{K_4^m} |w_m - \bar{w}_0|^2 \, dxdt \to 0 \text{ as } m \to \infty. \tag{3.176}$$

We take a cut-off functions $\phi = \phi(x) \in C_0^\infty(B_5)$, $\varrho = \varrho(t) \in C_0^\infty(-5^2, 5^2)$ such that

$$\begin{cases} 0 \le \phi \le 1, \ \phi = 1 \text{ on } B_3, \ \operatorname{supp} \phi \subset B_4, \ |D\phi| \le C, \\ 0 \le \varrho \le 1, \ \varrho = 1 \text{ on } (-3^2, 3^2), \ \operatorname{supp} \varrho \subset (-4^2, 4^2), \ |\varrho'| \le C. \end{cases} \tag{3.177}$$

Test (3.163) by $(w_m - \bar{w}_0)\phi^2\varrho$ to find that

$$\int_{K_5^m} (w_m)_t (w_m - \bar{w}_0)\phi^2\varrho + \langle \bar{a}(Dw_m, x^1), D[(w_m - \bar{w}_0)\phi^2\varrho] \rangle \, dxdt = 0. \tag{3.178}$$

Then a direct calculation gives

$$\begin{aligned} &\int_{K_5^m} \langle \bar{a}(Dw_m, x^1), D(w_m - \bar{w}_0) \rangle \phi^2 \varrho \, dxdt \\ &= \int_{K_5^m} \langle \bar{a}(Dw_m, x^1), D[(w_m - \bar{w}_0)\phi^2\varrho] \rangle \, dxdt \\ &\quad - \int_{K_5^m} \langle \bar{a}(Dw_m, x^1), 2\phi D\phi \rangle (w_m - \bar{w}_0)\varrho \, dxdt \\ &= -A_1 - A_3. \end{aligned} \tag{3.179}$$

We claim that

$$\begin{aligned} &\int_{K_5^m} \langle \bar{a}(Dw_m, x^1), D(w_m - \bar{w}_0) \rangle \phi^2 \varrho \, dxdt \\ &\quad = -A_1 - A_3 \to 0 \text{ as } m \to \infty. \end{aligned} \tag{3.180}$$

From (3.178) and (3.179), we have

$$
\begin{aligned}
A_1 &= \int_{K_5^m} (w_m)_t \phi^2 \varrho (w_m - \bar{w}_0)\, dxdt \\
&= \int_{K_5^m} (w_m)_t \phi^2 \varrho w_m \, dxdt \\
&\quad - \int_{K_5^m} ([w_m - \bar{w}_0])_t \phi^2 \varrho \bar{w}_0 \, dxdt - \int_{K_5^m} (\bar{w}_0)_t \phi^2 \bar{w}_0 \, dxdt \\
&= A_{11} - A_{12} - A_{13}.
\end{aligned} \tag{3.181}
$$

Since $\phi^2 \varrho \bar{w}_0 \in L^2(-4^2, 4^2; W^{1,2}(Q_5^+))$, we have from (3.170) that

$$
\begin{aligned}
A_{12} &= \int_{K_5^m} (w_m - \bar{w}_0)_t \phi^2 \varrho \bar{w}_0 \, dxdt \\
&= \int_{Q_5^+} (w_m - \bar{w}_0)_t \phi^2 \varrho \bar{w}_0 \, dxdt \to 0 \text{ as } m \to \infty.
\end{aligned} \tag{3.182}
$$

We next estimate $A_{11} - A_{13}$. By using Steklov average formulation and (3.177), we have

$$
\begin{aligned}
&\int_{-3^2}^{3^2} \varrho(t) \left(\int_{\Omega_4^m} ([w_m]_l)_t [w_m]_l |\phi(x)|^2 \, dx \right) dt \\
&= \frac{1}{2} \int_{-3^2}^{3^2} \varrho(t) \left(\frac{d}{dt} \| [w_m]_l(\cdot, t) \phi(\cdot) \|_{L^2(\Omega_4^m)}^2 \right) dt \\
&= \frac{1}{2} \int_{-3^2}^{3^2} \frac{d}{dt} \left(\varrho(t) \| [w_m]_l(\cdot, t) \phi(\cdot) \|_{L^2(\Omega_4^m)}^2 \right) dt \\
&\quad - \frac{1}{2} \int_{K_4^m} \varrho' ([w_m]_l)^2 \phi^2 dxdt \\
&= -\frac{1}{2} \int_{K_4^m} \varrho' ([w_m]_l)^2 \phi^2 \, dxdt.
\end{aligned} \tag{3.183}
$$

By sending $l \to 0$, we have from (3.181) and (3.183) that

$$
A_{11} = -\frac{1}{2} \int_{K_4^m} \varrho' w_m^2 \phi^2 \, dxdt. \tag{3.184}
$$

Similarly, we have

$$A_{13} = -\frac{1}{2}\int_{K_4^m} \varrho' \bar{w}_0^2 \phi^2 \, dxdt. \tag{3.185}$$

Thus from (3.184) and (3.185), we have

$$|A_{11} - A_{13}| \leq \int_{K_4^m} |\varrho'||w_m^2 - \bar{w}_0^2|\phi^2 \, dxdt. \tag{3.186}$$

We see from (3.170) and the weak lower semi-continuity that

$$\int_{K_4^m} |w_m|^2 + |\bar{w}_0|^2 \, dxdt \leq C. \tag{3.187}$$

By using (3.176) and (3.187), we have from (3.186) and Hölder's inequality that

$$\begin{aligned} |A_{11} - A_{13}| &\leq C\int_{K_4^m} |w_m^2 - \bar{w}_0^2| \, dxdt \\ &\leq C\left(\int_{K_4^m} |w_m - \bar{w}_0|^2 \, dxdt\right)^{\frac{1}{2}} \left(\int_{K_4^m} |w_m|^2 + |\bar{w}_0|^2 \, dxdt\right)^{\frac{1}{2}} \\ &\to 0 \text{ as } m \to \infty. \end{aligned} \tag{3.188}$$

To estimate A_3, we first apply Hölder's inequality, and then use the growth condition (3.2), the weak lower semi-continuity (3.172), (3.176) and (3.177) to find that

$$\begin{aligned} |A_3| &= \left|\int_{K_5^m} \langle \bar{a}(Dw_m, x^1), 2\phi D\phi\rangle (w_m - \bar{w}_0)\varrho \, dxdt\right| \\ &\leq C\left[\int_{K_5^m} |Dw_m|^2|D\phi|^2 dxdt\right]^{\frac{1}{2}} \left[\int_{K_5^m} |w_m - \bar{w}_0|^2\phi^2\varrho^2 \, dxdt\right]^{\frac{1}{2}} \\ &\leq C\left[\int_{K_4^m} |w_m - \bar{w}_0|^2 \, dxdt\right]^{\frac{1}{2}} \\ &\to 0 \text{ as } m \to \infty. \end{aligned} \tag{3.189}$$

By combining (3.182), (3.188) and (3.189), we see that the claim (3.180) holds. From (3.164) and (3.170), one can easily obtain that

$$D\bar{w}_m \rightharpoonup D\bar{w}_0 \text{ weakly in } L^2(Q_5). \tag{3.190}$$

Also from (3.180), we have

$$\int_{Q_5} \langle \bar{a}(D\bar{w}_m, x^1), D(\bar{w}_m - \bar{w}_0)\rangle \phi^2 \varrho \, dxdt \to 0 \text{ as } m \to \infty. \tag{3.191}$$

From (3.190), we have

$$\int_{Q_5} \langle \bar{a}(D\bar{w}_0, x^1), D(\bar{w}_m - \bar{w}_0)\rangle \phi^2 \varrho \, dxdt \to 0 \text{ as } m \to \infty. \tag{3.192}$$

By combining (3.191) and (3.192), the ellipticity condition (3.2) and the choice of cut-off function (3.177) imply that

$$\begin{aligned}
&\int_{Q_3} |D(\bar{w}_m - \bar{w}_0)|^2 \, dxdt \\
&\quad \leq \int_{Q_5} |D(\bar{w}_m - \bar{w}_0)|^2 \phi^2 \varrho \, dxdt \\
&\quad \leq C \int_{Q_5} \langle \bar{a}(D\bar{w}_m, x^1) - \bar{a}(D\bar{w}_0, x^1), D(\bar{w}_m - \bar{w}_0)\rangle \phi^2 \varrho \, dxdt \\
&\quad \to 0 \text{ as } m \to \infty.
\end{aligned} \tag{3.193}$$

Thus we reach a contradiction by comparing (3.166), (3.167) and (3.168) to (3.171), (3.172) and (3.193). This completes the proof. □

3.5 Global estimates in Reifenberg flat domains

We will establish global Calderón-Zygmund type estimates for the parabolic problem in Reifenberg flat domains. Recall from Remark 3.1.2 that the weak solution of u (3.3) can be extended to $\Omega \times \mathbb{R}$ satisfying

$$\begin{cases} u_t - \operatorname{div} a(Du, x, t) = \operatorname{div} F & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \end{cases} \tag{3.194}$$

where $F \in L^2(\Omega \times \mathbb{R}; \mathbb{R}^n)$. So in this section, we don't consider boundary estimates on the top and bottom parts, say $\Omega \times \{0\}$ and $\Omega \times \{t\}$. And we only obtain interior estimates and boundary estimates on the lateral boundary.

Lemma 3.5.1. *There exists a constant $N_1 = N_1(n, \lambda, \Lambda)$ such that the following holds. For any $\epsilon > 0$, one can find a small $\delta = \delta(\epsilon, n, \lambda, \Lambda) > 0$ so that for such $\delta > 0$, if $(\Omega \times \mathbb{R}, a(\xi, x, t))$ is $(\delta, 6)$-vanishing of codimension 1 and $B_9 \subset \Omega$, then for any weak solution u of (3.3) and*

$$\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) \leq 1\} \cap \{(x,t) \in \Omega_T : \mathcal{M}(|F|^2) \leq \delta^2\} \cap Q_1 \neq \emptyset, \tag{3.195}$$

we have

$$|\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) > N_1^2\} \cap Q_1| < \epsilon |Q_1|.$$

Proof. From (3.195), we see that there is a point $(y, s) \in Q_1$ such that

$$\frac{1}{|Q_\rho|} \int_{Q_\rho(y,s)} |Du|^2 \, dxdt \leq 1 \quad \text{ and } \quad \frac{1}{|Q_\rho|} \int_{Q_\rho(y,s)} |F|^2 \, dxdt \leq \delta^2, \tag{3.196}$$

for any $\rho > 0$. From the definition of $(\delta, 6)$-vanishing of codimension 1, $B_9 \subset \Omega$ implies that there exists a coordinate system such that

$$\fint_{Q_6} |\theta(a, \bar{a})|^2 \, dxdt \leq \delta^2, \tag{3.197}$$

for some $\bar{a}(\xi, x^1)$ satisfying (3.1) and (3.2). Since $(y, s) \in Q_1$, we have $Q_6 \subset Q_8(y, s)$. Thus we have from (3.196) that

$$\begin{cases} \dfrac{1}{|Q_6|} \displaystyle\int_{Q_6} |Du|^2 \, dxdt \leq \dfrac{|Q_8|}{|Q_6|} \dfrac{1}{|Q_8|} \displaystyle\int_{Q_8(y,s)} |Du|^2 \, dxdt \leq \left(\dfrac{4}{3}\right)^{n+2}, \\ \dfrac{1}{|Q_6|} \displaystyle\int_{Q_6} |F|^2 \, dxdt \leq \dfrac{|Q_8|}{|Q_6|} \dfrac{1}{|Q_8|} \displaystyle\int_{Q_8(y,s)} |F|^2 \, dxdt \leq \left(\dfrac{4}{3}\right)^{n+2} \delta^2. \end{cases} \tag{3.198}$$

In view of (3.198), (3.197) and Lemma 3.2.4, we apply Lemma 3.4.2 when u is replaced by $\left(\frac{3}{4}\right)^{n+2} u$ and F is replaced by $\left(\frac{3}{4}\right)^{n+2} F$ to find out that for any $\tau > 0$, there exist a small $\delta = \delta(\tau, n, \lambda, \Lambda) > 0$ and a weak solution w of

$$w_t - \operatorname{div} \bar{a}(Dw, x^1) = 0 \quad \text{ in } Q_4,$$

such that

$$\int_{Q_3} |D(u-w)|^2 \, dxdt \le \tau^2 \quad \text{ and } \|Dw\|_{L^\infty(Q_3)} \le n_1, \tag{3.199}$$

where $n_1 = n_1(n, \lambda, \Lambda)$ is a universal constant. We claim that

$$\begin{aligned} &\{(x,t) \in Q_1 : \mathcal{M}(|Du|^2) > N_1^2\} \\ &\qquad \subset \{(x,t) \in Q_1 : \mathcal{M}_{Q_2}(|D(u-w)|^2) > n_1^2\}, \end{aligned} \tag{3.200}$$

for $N_1^2 = \max\{4n_1^2, 3^{n+2}\}$. To do this, suppose that

$$(x_1, t_1) \in \{(x,t) \in Q_1 : \mathcal{M}_{Q_2}(|D(u-w)|^2) \le n_1^2\},$$

which implies

$$\frac{1}{|Q_\rho|} \int_{Q_\rho(x_1,t_1)} |D(u-w)|^2 \, dxdt \le n_1^2. \tag{3.201}$$

For $r \le 1$, $Q_r(x_1, t_1) \subset Q_2$, we have from (3.199) and (3.201) that

$$\begin{aligned} \frac{1}{|Q_r|} \int_{Q_r(x_1,t_1)} |Du|^2 dxdt &\le \frac{2}{|Q_r|} \int_{Q_r(x_1,t_1)} |D(u-w)|^2 + |Dw|^2 dxdt \\ &\le 4n_1^2. \end{aligned} \tag{3.202}$$

For $r > 1$, (3.196) and $Q_r(x_1, t_1) \subset Q_{3r}(y, s)$ imply that

$$\frac{1}{|Q_r|} \int_{Q_r(x_1,t_1)} |Du|^2 \, dxdt \le \frac{3^{n+2}}{|Q_{3r}|} \int_{Q_{3r}(y,s)} |Du|^2 \, dxdt \le 3^{n+2}. \tag{3.203}$$

Thus we have $(x_1, t_1) \in \{(x,t) \in Q_1 \ : \ \mathcal{M}(|Du|^2) > N_1^2\}$, and the claim (3.200) holds. From weak 1-1 estimate (3.2.8), (3.199) and (3.200), we obtain finally that

$$\begin{aligned} &|\{x \in Q_1 \ : \ \mathcal{M}(|Du|^2) > N_1^2\}| \\ &\quad \le |\{x \in Q_1 \ : \ \mathcal{M}_{Q_2}(|D(u-w)|^2) > n_1^2| \\ &\quad \le \frac{C}{n_1^2} \int_{Q_2} |D(u-w)|^2 \, dxdt \\ &\quad \le \frac{C\tau^2}{n_1^2} \\ &\quad < \epsilon |Q_1|, \end{aligned} \tag{3.204}$$

by taking $\tau = \tau(\epsilon, n, \lambda, \Lambda) > 0$ and the corresponding $\delta = \delta(\tau, n, \lambda, \Lambda) > 0$ satisfying the last inequality above. This completes our proof. $\square$

By using a scaling argument on Lemma 3.5.1, we have the next lemma.

Lemma 3.5.2. *There exists a constant $N_1 = N_1(n, \lambda, \Lambda)$ such that the following holds. For any $\epsilon > 0$, one can find a small $\delta = \delta(\epsilon, n, \lambda, \Lambda) > 0$ so that for such $\delta > 0$, if $(\Omega \times \mathbb{R}, a(\xi, x, t))$ is $(\delta, 6)$-vanishing of codimension 1 and $B_{9r} \subset \Omega$ with $r \in (0, 1]$, then for any weak solution u of (3.3) and*

$$\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) \le 1\} \cap \{(x,t) \in \Omega_T : \mathcal{M}(|F|^2) \le \delta^2\} \cap Q_r \neq \emptyset,$$

we have

$$|\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) > N_1^2\} \cap Q_r| < \epsilon |Q_r|.$$

By combining (3.148), (3.160), Lemma 3.4.3 and Lemma 3.4.4, we have the following lemma. Since our problem is invariant under translation, we may assume that the origin $\mathbf{0}$ is a point at the boundary.

Lemma 3.5.3. *There exists a constant $N_2 = N_2(n, \lambda, \Lambda)$ such that the following holds. For any $\epsilon > 0$, one can find a small $\delta = \delta(\epsilon, n, \lambda, \Lambda) > 0$ such that for such $\delta > 0$, if $(\Omega \times \mathbb{R}, a(\xi, x, t))$ is $(\delta, 6)$-vanishing of codimension 1 and $\mathbf{0} \in \partial\Omega$, then for any weak solution u of (3.3) and*

$$\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) \le 1\} \cap \{(x,t) \in \Omega_T : \mathcal{M}(|F|^2) \le \delta^2\} \cap Q_1 \neq \emptyset, \tag{3.205}$$

we have

$$|\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) > N_2^2\} \cap Q_1| < \epsilon |Q_1|. \tag{3.206}$$

Proof. From the definition of $(\delta, 6)$-vanishing of codimension 1, we have a coordinate system such that $\mathbf{0}$ is $-6\delta e_1$,

$$Q_6^+ \subset K_6 = \Omega_T \cap Q_6 \subset Q_6 \cap \{x : x^1 > -12\delta\}, \tag{3.207}$$

and

$$\fint_{Q_6} |\theta(a; Q_6)|^2 \, dxdt \le \delta^2. \tag{3.208}$$

for some $\bar{a}(\xi, x^1)$ satisfying (3.1) and (3.2).

From (3.205), there is a point $(y, s) \in Q_1(-6\delta e_1, 0) \subset Q_2$ such that

$$(y,s) \in \{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) \le 1\} \cap \{(x,t) \in \Omega_T : \mathcal{M}(|F|^2) \le \delta^2\} \cap Q_2, \tag{3.209}$$

which implies

$$\frac{1}{|Q_\rho|}\int_{K_\rho(y,s)}|Du|^2\,dx\le 1 \quad \text{and}\ \frac{1}{|Q_\rho|}\int_{K_\rho(y,s)}|F|^2\,dx\le\delta^2, \tag{3.210}$$

for any $\rho>0$.

Since $(y,s)\in Q_2$, we have $Q_6\subset Q_8(y,s)$. Thus we have from (3.207) that

$$K_6\subset K_8(y,s) \quad \text{and} \quad |Q_8|\le 2\cdot\left(\frac{4}{3}\right)^{n+2}|K_6|. \tag{3.211}$$

From (3.211), we have

$$\begin{cases} \dfrac{1}{|K_6|}\displaystyle\int_{K_6}|Du|^2\,dx\le\dfrac{|Q_8|}{|K_6|}\dfrac{1}{|Q_8|}\displaystyle\int_{K_8(y,s)}|Du|^2\,dx\le 2\cdot\left(\dfrac{4}{3}\right)^{n+2}, \\ \dfrac{1}{|K_6|}\displaystyle\int_{K_6}|F|^2\,dx\le\dfrac{|Q_8|}{|K_6|}\dfrac{1}{|Q_8|}\displaystyle\int_{K_8(y,s)}|F|^2\,dx\le 2\cdot\left(\dfrac{4}{3}\right)^{n+2}\delta^2. \end{cases} \tag{3.212}$$

To apply Lemma 3.4.4, we use the following normalization from Lemma 3.2.4. Set $\sigma_3^2=2\cdot\left(\frac{4}{3}\right)^{n+2}>1$. Then by setting

$$\begin{cases} \tilde{u}=\sigma_3^{-1}u, & \tilde{F}=\sigma_3^{-1}F, \\ \tilde{a}(\xi,x,t)=\sigma_3^{-1}a(\sigma_3\xi,x,t), & \hat{a}(\xi,x^1)=\sigma_3^{-1}\bar{a}(\sigma_3\xi,x^1), \end{cases} \tag{3.213}$$

we find out from (3.3), (3.208), (3.212) and Lemma 3.2.4 that

$$\begin{cases} \tilde{u}_t-\operatorname{div}\tilde{a}(D\tilde{u},x,t) &= \operatorname{div}\tilde{F} & \text{in } K_6, \\ \tilde{u} &= 0 & \text{on } \partial_w K_6, \end{cases} \tag{3.214}$$

$$\begin{cases} \dfrac{1}{|K_6|}\displaystyle\int_{K_6}|D\tilde{u}|^2\,dx\le\dfrac{\sigma_3^{-2}}{|K_6|}\displaystyle\int_{K_6}|Du|^2\,dx\le 1, \\ \dfrac{1}{|K_6|}\displaystyle\int_{K_6}|\tilde{F}|^2\,dx\le\dfrac{\sigma_3^{-2}}{|K_6|}\displaystyle\int_{K_6}|F|^2\,dx\le\delta^2, \end{cases} \tag{3.215}$$

and

$$⨍_{Q_6}|\theta(\tilde{a},\hat{a})|^2\,dxdt\le\delta^2. \tag{3.216}$$

By using (3.214), (3.215) and (3.216), we apply Lemma 3.4.4 to find that for any $\tau > 0$, there exist a small $\delta = \delta(\tau, n, \lambda, \Lambda) > 0$ and $\tilde{h}$ such that

$$\int_{K_3} |D(\tilde{u} - \tilde{h})|^2 \, dxdt \leq \sigma_3^{-2} \tau^2 \quad \text{and} \quad \|D\tilde{h}\|_{L^\infty(B_3)} \leq n_2. \tag{3.217}$$

Set $h = \sigma_3 \tilde{h}$. Then we have from (3.217) that

$$\int_{K_3} |D(u - h)|^2 \, dxdt \leq \tau^2 \quad \text{and} \quad \|Dh\|_{L^\infty(B_3)} \leq \sigma_3 n_2, \tag{3.218}$$

where $n_2 = n_2(n, \lambda, \Lambda)$ is a universal constant.

Now, we claim that

$$\begin{aligned} &\{(x,t) \in Q_2 : \mathcal{M}(|Du|^2) > N_2^2\} \\ &\qquad \subset \{(x,t) \in Q_2 : \mathcal{M}_{K_3}(|D(u-h)|^2) > n_2^2\}, \end{aligned} \tag{3.219}$$

for $N_2^2 = \max\{4\sigma_3^2 n_2^2, 4^{n+2}\}$. To do this, suppose that

$$(x_1, t_1) \in \{x \in Q_2 : \mathcal{M}_{K_3}(|D(u-h)|^2) \leq n_2^2\}. \tag{3.220}$$

From the fact that $(x_1, t_1) \in Q_2$, we have

$$Q_\rho(x_1, t_1) \subset Q_3 \quad (0 < \rho < 1), \tag{3.221}$$

and (3.220) implies that

$$\frac{1}{|Q_\rho|} \int_{K_\rho(x_1,t_1)} |D(u-h)|^2 \, dxdt \leq n_2^2 \quad (0 < \rho < 1). \tag{3.222}$$

If $0 < r \leq 1$, then from (3.218), (3.222) and the fact that $K_r(x_1, t_1) \subset K_3$, we have

$$\begin{aligned} \frac{1}{|Q_r|} \int_{K_r(x_1,t_1)} |Du|^2 dxdt &\leq \frac{2}{|Q_r|} \int_{K_r(x_1,t_1)} |D(u-h)|^2 + |Dh|^2 dxdt \\ &\leq 4\sigma_3^2 n_2^2. \end{aligned} \tag{3.223}$$

If $r > 1$, then from $(x_1, t_1) \in Q_{2r} \subset Q_{3r}(y, s)$ and the fact that (3.210), we have

$$\frac{1}{|Q_r|}\int_{K_r(x_1,t_1)} |Du|^2\, dxdt \le \frac{4^{n+2}}{|Q_{4r}|}\int_{K_{4r}(y,s)} |Du|^2\, dxdt \le 4^{n+2}. \tag{3.224}$$

We see from (3.223) and (3.224) that $(x_1,t_1) \in \{(x,t) \in Q_2 : \mathcal{M}(|Du|^2) \le N_2^2\}$ and the claim (3.219) holds.

From weak 1-1 estimate (3.2.8), (3.218) and (3.219), we finally have

$$\begin{aligned}
&|\{x \in Q_2 \ : \ \mathcal{M}(|Du|^2) > N_2^2\}| \\
&\quad \le |\{x \in Q_2 \ : \ \mathcal{M}_{K_3}(|D(u-w)|^2) > n_2^2| \\
&\quad \le \frac{C}{n_2^2}\int_{K_3} |D(u-w)|^2\, dxdt \\
&\quad \le \frac{C\tau^2}{n_2^2} \\
&\quad < \epsilon|Q_2|,
\end{aligned} \tag{3.225}$$

by taking $\tau = \tau(\epsilon, n, \lambda, \Lambda) > 0$ and the corresponding $\delta = \delta(\tau, n, \lambda, \Lambda) > 0$ satisfying the last inequality above. From the choice of the coordinate system, we have $-6\delta e_1 \in \partial\Omega$ and $Q_1(-6\delta e_1) \subset Q_2$. We transform the coordinate back so that $\mathbf{0} \in \partial\Omega$. Then we see from (3.225) that the lemma holds. □

By using a scaling argument, we have the following lemma.

Lemma 3.5.4. *There exists a constant $N_2 = N_2(n, \lambda, \Lambda)$ such that the following holds. For any $\epsilon > 0$, one can find a small $\delta = \delta(\epsilon, n, \lambda, \Lambda) > 0$ so that for such δ if $(\Omega \times \mathbb{R}, a(\xi, x, t))$ is $(\delta, 6)$-vanishing of codimension 1, $\mathbf{0} \in \partial\Omega$ and $r \in (0, 1]$, then for any weak solution u of (3.3) and*

$$\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) \le 1\} \cap \{(x,t) \in \Omega_T : \mathcal{M}(|F|^2) \le \delta^2\} \cap Q_r \ne \emptyset,$$

we have

$$|\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) > N_2^2\} \cap Q_r| < \epsilon|Q_r|.$$

Take $N_0 = \max\{N_1, N_2, 1\}$ from N_1 in Lemma 3.5.2 and N_2 in Lemma 3.5.4. From Lemma 3.5.2 and Lemma 3.5.4, we have the following lemma.

Lemma 3.5.5. *There exists a constant $N_0 = N_0(n, \lambda, \Lambda) \ge 1$ such that the following holds. For any $\epsilon > 0$, one can find a small $\delta = \delta(\epsilon, n, \lambda, \Lambda) > 0$*

so that for such δ, if $(\Omega \times \mathbb{R}, a(\xi, x, t))$ is $(\delta, 120)$-vanishing of codimension 1 and $r \in (0, 1]$, then for any weak solution u of (3.3) and

$$|\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) > N_0^2\} \cap Q_r| \geq \epsilon |Q_r|, \tag{3.226}$$

then

$$K_r \subset \{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) > 1\} \cup \{(x,t) \in \Omega_T : \mathcal{M}(|F|^2) > \delta^2\}. \tag{3.227}$$

Proof. We prove this lemma by contradiction. If Q_r satisfies (3.226) but (3.227) is false, then there exists $(y, s) \in Q_r$ such that

$$(y,s) \in \{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) \leq 1\} \cap \{(x,t) \in \Omega_T : \mathcal{M}(|F|^2) \leq \delta^2\} \cap Q_r. \tag{3.228}$$

If $B_{9r} \subset \Omega$, then this contradicts Lemma 3.5.2. Thus Lemma 3.5.5 holds when $B_{9r} \subset \Omega$.

So suppose that there exists a point such that

$$x_0 \in B_{9r} \cap \partial\Omega. \tag{3.229}$$

Since $x_0 \in B_{9r}$, we have $Q_r \subset Q_{10r}(x_0, t_0)$ which implies

$$\begin{aligned} &|\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) > N_2^2\} \cap Q_r(x_0, 0)| \\ &\quad \leq |\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) > N_2^2\} \cap Q_{10r}(x_0, 0)|. \end{aligned} \tag{3.230}$$

To apply Lemma 3.5.4, we use a scaling from Lemma 3.2.4 by setting

$$\begin{cases} \tilde{u}(x,t) = \dfrac{u(x_0 + 20x, 20^2 t)}{20}, & \tilde{F}(x,t) = \dfrac{F(x_0 + 20x, 20^2 t)}{20}, \\ \tilde{a}(\xi, x, t) = a(\xi, x_0 + 20x, 20^2 t), & \tilde{\Omega} = \left\{ \dfrac{x - x_0}{20} : x \in \Omega \right\}. \end{cases} \tag{3.231}$$

Then from Lemma 3.2.4 and the fact that $(\Omega \times \mathbb{R}, a(\xi, x, t))$ is $(\delta, 120)$-vanishing with codimension 1 for λ and Λ, we have

$$(\tilde{\Omega} \times \mathbb{R}, \tilde{a}(\xi, x, t)) \text{ is } (\delta, 6)\text{-vanishing with codimension 1 for } \lambda \text{ and } \Lambda, \tag{3.232}$$

and from Lemma 3.2.4, we see that $\tilde{u}$ is a weak solution of

$$\begin{cases} \tilde{u}_t - \text{div } \tilde{a}(D\tilde{u}, x, t) & = & \text{div } \tilde{F} & \text{in } \tilde{\Omega}_{\frac{T}{20^2}}, \\ \tilde{u} & = & 0 & \text{on } \partial_p \tilde{\Omega}_{\frac{T}{20^2}}. \end{cases} \tag{3.233}$$

Also from (3.229) and (3.231), we have $Q_{10r}(x_0, 0) \subset Q_{20r}$ and

$$\begin{aligned} &|\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) > N_2^2\} \cap Q_{10r}(x_0, 0)| \\ &\quad \le 20^{n+2} |\{(x,t) \in \tilde{\Omega}_{\frac{T}{20^2}} : \mathcal{M}(|D\tilde{u}|^2) > N_2^2\} \cap Q_r|. \end{aligned} \tag{3.234}$$

Since $(y, s) \in Q_r \subset Q_{10r}(x_0, 0)$, we have $(y - x_0, s) \in Q_{20r}$ and $\left(\frac{y-x_0}{20}, \frac{s}{20^2}\right) \in Q_r$. Thus from (3.228) and (3.231), we see that

$$\begin{aligned} \left(\frac{y - x_0}{20}, \frac{s}{20^2}\right) &\in \{(x,t) \in \tilde{\Omega}_{\frac{T}{20^2}} : \mathcal{M}(|D\tilde{u}|^2) \le 1\} \\ &\qquad \cap \{(x,t) \in \tilde{\Omega}_{\frac{T}{20^2}} : \mathcal{M}(|\tilde{F}|^2) \le \delta^2\} \cap Q_r. \end{aligned} \tag{3.235}$$

In view of (3.232), (3.233) and (3.235), we use Lemma 3.5.4 to find that

$$|\{(x,t) \in \tilde{\Omega}_{\frac{T}{20^2}} : \mathcal{M}(|D\tilde{u}|^2) > N_2^2\} \cap Q_r| \le \frac{\epsilon}{20^{n+2}} |Q_r|. \tag{3.236}$$

Thus by using (3.232) and (3.236), we see that

$$\begin{aligned} &|\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) > N_2^2\} \cap Q_r| \\ &\quad \le |\{(x,t) \in \Omega_T : \mathcal{M}(|Du|^2) > N_2^2\} \cap Q_{10r}(x_0, 0)| \\ &\quad \le 20^{n+2} |\{(x,t) \in \tilde{\Omega}_{\frac{T}{20^2}} : \mathcal{M}(|D\tilde{u}|^2) > N_2^2\} \cap Q_r| \\ &\quad < \epsilon |Q_r|. \end{aligned} \tag{3.237}$$

Since $N_0 = \max\{N_1, N_2, 1\}$, (3.237) contradicts (3.226). This finishes the proof. $\square$

Lemma 3.5.6. *Under the same assumptions in Lemma 3.5.5, we further assume*

$$|\{(x,t) \in \Omega : \mathcal{M}(|Du|^2) > N_0^2\}| < \epsilon |Q_1|. \tag{3.238}$$

Then we have

$$
\begin{aligned}
&|\{(x,t)\in\Omega_T : \mathcal{M}(|Du|^2) > N_0^{2k}\}| \\
&\quad \le \sum_{i=1}^{k} \epsilon_1^i |\{(x,t)\in\Omega_T : \mathcal{M}(|F|^2) > \delta^2 N_0^{2(k-i)}\}| \\
&\qquad + \epsilon_1^k |\{(x,t)\in\Omega_T : \mathcal{M}(|Du|^2) > 1\}|.
\end{aligned}
$$

Proof. We prove by induction on k. The case $k = 1$ follows from Lemma 3.2.9 and Lemma 3.5.5 when

$$
\begin{aligned}
&E = \{(x,t)\in\Omega_T : \mathcal{M}(|Du|^2) > N_0^2\}, \\
&F = \{(x,t)\in\Omega_T : \mathcal{M}(|F|^2) > \delta^2 \text{ or } \mathcal{M}(|Du|^2) > 1\}.
\end{aligned}
$$

Suppose that the conclusion is true for $k \ge 2$. By using (3.2.4), we normalize u by $u_{N_0} = u/N_0$ and F by $F_{N_0} = F/N_0$, respectively, to see from Lemma 3.5.5 that

$$
\begin{aligned}
&|\{(x,t)\in\Omega_T : \mathcal{M}(|Du_{N_0}|^2) > N_0^2\}| \\
&\quad = |\{(x,t)\in\Omega_T : \mathcal{M}(|Du|^2) > N_0^4\}| \\
&\quad \le |\{(x,t)\in\Omega_T : \mathcal{M}(|Du|^2) > N_0^2\}| \\
&\quad < \epsilon|B_1|.
\end{aligned}
\tag{3.239}
$$

Then using the induction assumption, we calculate as follows:

$$
\begin{aligned}
&|\{(x,t)\in\Omega_T : \mathcal{M}(|Du|^2) > N_0^{2(k+1)}\}| \\
&\quad = |\{(x,t)\in\Omega_T : \mathcal{M}(|Du_{N_0}|^2) > N_0^{2k}\}| \\
&\quad \le \sum_{i=1}^{k} \epsilon_1^i |\{(x,t)\in\Omega_T : \mathcal{M}(|F_{N_0}|^2) > \delta^2 N_0^{2(k-i)}\}| \\
&\qquad + \epsilon_1^k |\{(x,t)\in\Omega_T : \mathcal{M}(|Du_{N_0}|^2) > 1\}| \\
&\quad \le \sum_{i=1}^{k+1} \epsilon_1^i |\{(x,t)\in\Omega_T : \mathcal{M}(|F|^2) > \delta^2 N_0^{2(k+1-i)}\}| \\
&\qquad + \epsilon_1^{k+1} |\{(x,t)\in\Omega_T : \mathcal{M}(|Du|^2) > 1\}|.
\end{aligned}
$$

as required. □

We are finally ready to prove Theorem 3.1.8.

Proof of Main Theorem. We first take $\epsilon = \epsilon(n, \lambda, \Lambda) > 0$ so that

$$N_0^p 20^{n+2}\epsilon \leq \frac{1}{2}, \tag{3.240}$$

and then find a corresponding $0 < \delta = \delta(\epsilon, n, \lambda, \Lambda) < \frac{1}{2}$ from Lemma 3.5.6.

We next recall Lemma 3.2.4 to consider

$$u_1 = \frac{\delta|\Omega_T|^{\frac{1}{p}} u}{\|F\|_{L^p(\Omega_T)}} \text{ and } F_1 = \frac{\delta|\Omega_T|^{\frac{1}{p}} F}{\|F\|_{L^p(\Omega_T)}}. \tag{3.241}$$

Then by standard L^2-estimate and Hölder's inequality, we have

$$\begin{aligned} \int_{\Omega_T} |Du_1|^2 dx &\leq C \int_{\Omega_T} |F_1|^2 \, dx \\ &\leq C \left(\int_{\Omega_T} |F_1|^p \, dx \right)^{\frac{2}{p}} \left(\int_{\Omega_T} 1 \, dx \right)^{\frac{p-2}{p}} \\ &\leq C\delta^2 |\Omega_T|. \end{aligned} \tag{3.242}$$

From (3.240) and weak 1-1 estimate Lemma 3.2.8, we have

$$\begin{aligned} |\{(x,t) \in \Omega_T : \mathcal{M}(|Du_1|^2) > N_0^2\}| &\leq C \int_{\Omega_T} |Du_1|^2 \, dx \\ &\leq C\delta^2 |\Omega_T| \\ &< \epsilon |Q_1|, \end{aligned} \tag{3.243}$$

by further selecting a smaller δ satisfying the last inequality. We set $\epsilon_1 = \left(\frac{10}{1-\delta}\right)^{n+2} \epsilon$. Then from (3.240), we have

$$N_0^p \epsilon_1 = N_0^p \left(\frac{10}{1-\delta}\right)^{n+2} \epsilon \leq N_0^p 20^{n+2} \epsilon \leq \frac{1}{2}. \tag{3.244}$$

From Lemma 3.2.8 and (3.241), we have

$$\|\mathcal{M}(\delta^{-2} F_1^2)\|_{L^{\frac{p}{2}}(\Omega_T)}^{\frac{p}{2}} \leq C \|\delta^{-2} F_1^2\|_{L^{\frac{p}{2}}(\Omega_T)}^{\frac{p}{2}} \leq C|\Omega_T|. \tag{3.245}$$

Thus (3.245) and Lemma 3.2.10 imply that

$$\begin{aligned}
&\sum_{k=i}^{\infty} N_0^{p(k-i)} |\{(x,t) \in \Omega_T : \mathcal{M}(|F_1|^2) > \delta^2 N_0^{2(k-i)}\}| \\
&\qquad \leq C \|\mathcal{M}(\delta^{-2} F_1^2)\|_{L^{\frac{p}{2}}(\Omega_T)}^{\frac{p}{2}} \\
&\qquad \leq C|\Omega_T|.
\end{aligned} \tag{3.246}$$

In view of (3.243), (3.244), (3.246), we use Lemma 3.5.6 to find that

$$\begin{aligned}
&\sum_{k=1}^{\infty} N_0^{pk} |\{(x,t) \in \Omega_T : \mathcal{M}(|Du_1|^2) > N_0^{2k}\}| \\
&\quad \leq \sum_{k=1}^{\infty} N_0^{pk} \sum_{i=1}^{k} \epsilon_1^i |\{(x,t) \in \Omega_T : \mathcal{M}(|F_1|^2) > \delta^2 N_0^{2(k-i)}\}| \\
&\quad\quad + \sum_{k=1}^{\infty} N_0^{pk} \epsilon_1^k |\{(x,t) \in \Omega_T : \mathcal{M}(|Du_1|^2) > 1\}| \\
&\quad = \sum_{i=1}^{\infty} [N_0^p \epsilon_1]^i \left[\sum_{k=i}^{\infty} N_0^{p(k-i)} |\{(x,t) \in \Omega_T : \mathcal{M}(|F_1|^2) > \delta^2 N_0^{2(k-i)}\}| \right] \\
&\quad\quad + \sum_{k=1}^{\infty} [N_0^p \epsilon_1]^k |\{(x,t) \in \Omega_T : \mathcal{M}(|Du_1|^2) > 1\}| \\
&\quad \leq C|\Omega_T| \sum_{k=1}^{\infty} [N_0^p \epsilon_1]^k \\
&\quad \leq C|\Omega_T|.
\end{aligned}$$

But then Lemma 3.2.8 and Lemma 3.2.10 imply $Du_1 \in L^p(\Omega_T; \mathbb{R}^n)$ with the estimate

$$\|Du_1\|_{L^p(\Omega_T;\mathbb{R}^n)}^p \leq C|\Omega_T|.$$

We recall the definition of u_1 and F_1 in (3.241) to conclude that

$$\|Du\|_{L^p(\Omega_T;\mathbb{R}^n)} \leq C\|F\|_{L^p(\Omega_T;\mathbb{R}^n)}.$$

the constant C depending only on n, λ, Λ, p and $|\Omega_T|$. $\square$

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국문초록

이 학위 논문에서는 매끄럽지 않은 영역에서 정의되고 측정가능한 비선형 계수 함수를 가지는 타원형 및 포물형 편미분 방정식들의 대역적 정칙성 가늠을 연구한다. 본 논문의 목적은 이러한 방정식에서 대역적 칼데론-지그먼드 이론이 성립하기 위한 최소 조건을 얻는것이다. 비선형 계수 함수가 한 변수에 대해서는 단지 측정가능하고, 다른 변수에 대해서는 BMO semi-norm을 만족하며, Reifenberg 센스로 영역이 편평한 경계를 가질때 비제차항과 약해의 그래디언트가 같은 적분 공간 안에 속한다는 것을 증명한다.

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학번: 2006-22887